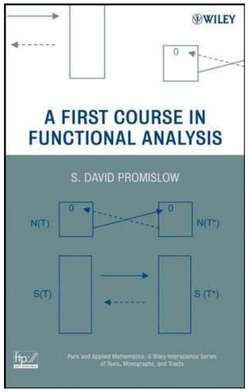
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems

3.4. Bounded Inverses—Proofs of Theorems



Theorem 3.6

Theorem 3.6. Given an injective $T \in \mathcal{B}(X, Y)$ for which both X and Y are Banach spaces, the following are equivalent:

(i) T^{-1} is bounded;

(ii) T is bounded below;

(iii) $R(T)$ (the range of T) is closed.

Proof. (i)⇒(ii). If T^{-1} is bounded and $\|T^{-1}\| = k < \infty$, then for any $x \in X$ with $\|x\| = 1$ we have by Note 2.4.A that

$$1 = \|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| = k\|Tx\|$$

and so $\|Tx\| \geq 1/k > 0$. So T is bounded below, as claimed.

Theorem 3.6 (continued 1)

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(i) T^{-1} is bounded;

(ii) T is bounded below;

(iii) $R(T)$ (the range of T) is closed.

Proof (continued). (ii)⇒(iii). For $y \in \overline{R(T)}$, choose $(x_n) \subseteq X$ such that (Tx_n) converges to y (such a sequence exists by Theorem 2.2.A(iii)). Then (Tx_n) is Cauchy and since (by Note 3.4.A) $\|Tx\| \geq k\|x\|$ for some $k > 0$, then for all $m, n \in \mathbb{N}$ we have $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \geq k\|x_n - x_m\|$ so that (x_n) is Cauchy in X and so convergent to some $x \in X$. But since T is bounded (and so continuous by Theorem 2.6), then $y = \lim(Tx_n) = T(\lim x_n) = Tx$. So $y \in R(T)$, $R(T) = \overline{R(T)}$, and $R(T)$ is closed, as claimed.

Theorem 3.6 (continued 2)

Theorem 3.6. Given an injective $T \in \mathcal{B}(X, Y)$ for which both X and Y are Banach spaces, the following are equivalent:

(i) T^{-1} is bounded;

(iii) $R(T)$ (the range of T) is closed.

Proof (continued). (iii)⇒(i). Suppose $R(T)$ is a closed subset of Y. Then $R(T)$ is a closed subspace of Y since $y_1, y_2 \in Y$ implies there exists $x_1, x_2 \in X$ with $T(x_1) = y_1$ and $T(x_2) = y_2$. So any linear combination of y_1 and y_2 is the image under T of the corresponding linear combination of x_1 and x_2 (say, $ay_1 + by_2 = T(ax_1 + bx_2)$ where $ax_1 + bx_2 \in X$). Then by Theorem 2.16, $R(T)$ is a Banach space itself. Let $U \subseteq X$ be open. Then $(T^{-1})^{-1}U = T(U)$ and by the Open Mapping Theorem (Theorem 3.5), $T(U)$ is open in $R(T)$ (since T is bounded and onto [surjective] its range $R(T)$). So inverse images of open sets (i.e., inverse images with respect to T^{-1}) are open in $R(T)$ and so T^{-1} is continuous (see Note 2.2.C) and hence, by Theorem 2.6, T^{-1} is bounded on $R(T)$ (the subset of Y on which T^{-1} is defined) and so (i) follows. □