Chapter 3. Major Banach Space Theorems

3.4. Bounded Inverses—Proofs of Theorems
Theorem 3.6. Given an injective \( T \in \mathcal{B}(X, Y) \) for which both \( X \) and \( Y \) are Banach spaces, the following are equivalent:

(i) \( T^{-1} \) is bounded;

(ii) \( T \) is bounded below;

(iii) \( R(T) \) (the range of \( T \)) is closed.

Proof of (i) \( \Rightarrow \) (ii). If \( T^{-1} \) is bounded and \( \|T^{-1}\| = k < \infty \), then for any \( x \in X \) with \( \|x\| = 1 \) we have

\[
1 = \|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| = k\|Tx\|
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and so \( \|Tx\| \geq 1/k > 0 \).
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**Proof of (ii) $\implies$ (iii).** For $y \in \overline{R(T)}$, choose $(x_n) \subseteq X$ such that $(Tx_n)$ converges to $y$ (such a sequence exists by Theorem 2.2.A(iii)). Then $(Tx_n)$ is Cauchy and since $\|Tx_n\| \geq k\|x_n\|$ for some $k > 0$ and for all $n \in \mathbb{N}$, then $(x_n)$ is Cauchy in $X$ and so convergent to some $x \in X$. But since $T$ is bounded (and so continuous by Theorem 2.6),

$$y = \lim(Tx_n) = T(\lim x_n) = Tx.$$ 

So $y \in R(T)$ and $R(T)$ is closed.
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**Proof of (iii)$\implies$(i).** Suppose $R(T)$ is a closed subset of $Y$. Then $R(T)$ is a closed subspace of $Y$ since $y_1, y_2 \in Y$ implies there exists $x_1, x_2 \in X$ with $T(x_1) = y_1$ and $T(x_2) = y_2$. So any linear combination of $y_1$ and $y_2$ is the image under $T$ of the corresponding linear combination of $x_1$ and $x_2$ (say, $ay_1 + by_2 = T(ax_1 + bx_2)$ where $ax_1 + bx_2 \in X$).
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Theorem 3.6, (iii) $\Rightarrow$ (i)

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