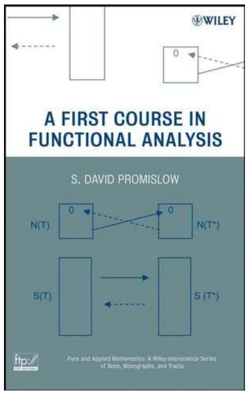
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems
3.5. Closed Linear Operators—Proofs of Theorems



Theorem 3.7

Theorem 3.7. If $T \in \mathcal{L}(X, Y)$ is injective (one to one) and closed, then T^{-1} is closed.

Proof. Suppose $T \in \mathcal{L}(X, Y)$ is injective and closed. Let $(y, x) \in \overline{G_{T^{-1}}}$. Then there is a sequence $((y_n, x_n))_{n=1}^{\infty} \subseteq G_{T^{-1}}$ that converges to (y, x) by Theorem 2.2.A(iii). Since we are using the sup norm in $X \times Y$ (and $Y \times X$) then $((x_n, y_n))_{n=1}^{\infty} \subseteq G_T$ converges to (x, y). Since T is closed (i.e., G_T is a closed set) then $(x, y) \in G_T$ by Theorem 2.2.A(iii) and so $y = Tx$. Then $x = T^{-1}y$ and so $(y, x) \in G_{T^{-1}}$. Therefore $\overline{G_{T^{-1}}} = G_{T^{-1}}$ and $G_{T^{-1}}$ is closed. That is, T^{-1} is closed. □

Lemma 3.5.A

Lemma 3.5.A. If $T \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces, is bounded then T is closed.

Proof. Let (x_n, y_n) be a sequence in G_T which converges under the sup norm to (x, y). Then $(x_n) \to x$ in X and $(y_n) \to y$ in Y. Since T is bounded, then it is continuous by Theorem 2.6. So

$$Tx = T(\lim x_n) = \lim(Tx_n) = \lim y_n = y.$$

So $(x, y) \in G_T$ and by Theorem 2.2.A(iii), G_T is closed. Hence, by definition, T is closed, as claimed. □

Theorem 3.9. Closed Graph Theorem

Theorem 3.9. Closed Graph Theorem.
If $T \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces, then T is closed if and only if it is bounded.

Proof. Lemma 3.5.A shows that if T is bounded then T is closed.

Now suppose T is closed. Let P_X be the projection $P_X : G_T \to X$ defined as $P_X(x, y) = x$, and let $P_Y : G_T \to Y$ be defined as $P_Y(x, y) = y$. Since $X \times Y$ has the sup norm, if $\|(x, y)\| = 1$ then $\|P_X(x, y)\| = \|x\| \le 1$. For $(x, 0)$ with $\|(x, 0)\| = \|x\| = 1$, we see that $\|P_X\| = 1$ and similarly $\|P_Y\| = 1$. So P_X and P_Y are bounded. Since X and Y are Banach spaces, then $X \times Y$ is a Banach space by Theorem 2.10.A. Since G_T is closed in $X \times Y$, then G_T is a Banach space by Theorem 2.16. The range of P_X is all of X since T is defined on X, and so $R(P_X) = X$ is closed. So by Theorem 3.6, P_X^{-1} is bounded. So $T = P_Y P_X^{-1}$ is bounded by Proposition 2.8. (Notice $P_X^{-1} : X \to G_T$ and $P_Y : G_T \to Y$, so $T = P_Y P_X^{-1}$.) □