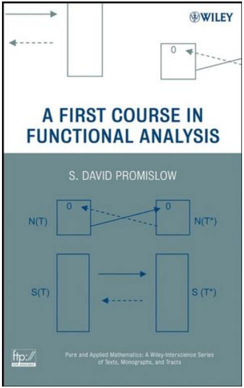
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems

3.5. Closed Linear Operators—Proofs of Theorems



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Lemma 3.5.A

Lemma 3.5.A. If $T \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces, is bounded then T is closed.

Proof. Let (x_n, y_n) be a sequence in G_T which converges under the sup norm to (x, y). Then $(x_n) \to x$ in X and $(y_n) \to y$ in Y. Since T is bounded, then it is continuous by Theorem 2.6. So

$$Tx = T(\lim x_n) = \lim(Tx_n) = \lim y_n = y.$$

So $(x, y) \in G_T$ and by Theorem 2.2.A(iii), G_T is closed. Hence, by definition, T is closed, as claimed. □

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