Chapter 3. Major Banach Space Theorems
3.5. Closed Linear Operators—Proofs of Theorems
1. Theorem 3.7

2. Lemma

3. Theorem 3.9, Closed Graph Theorem
Theorem 3.7

**Theorem 3.7.** If \( T \in \mathcal{L}(X, Y) \) is injective (one to one) and closed, then \( T^{-1} \) is closed.

**Proof.** Suppose \( T \in \mathcal{L}(X, Y) \) is injective and closed.
Theorem 3.7. If $T \in \mathcal{L}(X, Y)$ is injective (one to one) and closed, then $T^{-1}$ is closed.

Proof. Suppose $T \in \mathcal{L}(X, Y)$ is injective and closed. Let $(y, x) \in \overline{G_T}$. Then there is a sequence $((y_n, x_n))_{n=1}^{\infty} \subseteq G_T^{-1}$ that converges to $(y, x)$ by Theorem 2.2.A(iii).
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Proof. Suppose $T \in \mathcal{L}(X, Y)$ is injective and closed. Let $(y, x) \in \overline{G_{T^{-1}}}$. Then there is a sequence $((y_n, x_n))_{n=1}^{\infty} \subseteq G_{T^{-1}}$ that converges to $(y, x)$ by Theorem 2.2.A(iii). Since we are using the sup norm in $X \times Y$ (and $Y \times X$) then $((x_n, y_n))_{n=1}^{\infty} \subseteq G_T$ converges to $(x, y)$. 
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**Theorem 3.7.** If $T \in \mathcal{L}(X, Y)$ is injective (one to one) and closed, then $T^{-1}$ is closed.

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Lemma. If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.

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Lemma. If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.

Proof. Let $(x_n, y_n)$ be a sequence in $G_T$ which converges under the sup norm to $(x, y)$. Then $(x_n) \to x$ in $X$ and $(y_n) \to y$ in $Y$. 
Lemma. If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.

Proof. Let $(x_n, y_n)$ be a sequence in $G_T$ which converges under the sup norm to $(x, y)$. Then $(x_n) \to x$ in $X$ and $(y_n) \to y$ in $Y$. Since $T$ is bounded, then it is continuous by Theorem 2.6.
Lemma. If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.

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$$Tx = T \left( \lim x_n \right) = \lim \left( Tx_n \right) = \lim y_n = y.$$
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So $(x, y) \in G_T$ and $G_T$ is closed.
**Lemma.** If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.

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Theorem 3.9, Closed Graph Theorem

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If $T \in \mathcal{L}(X, Y)$ where $X$ and $Y$ are Banach spaces, then $T$ is closed if and only if it is bounded.

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Suppose $T$ is closed.

Let $P_X: G_T \rightarrow X$ be defined as $P_X(x, y) = x$, and let $P_Y: G_T \rightarrow Y$ be defined as $P_Y(x, y) = y$.

Since $X \times Y$ has the sup norm, if $\| (x, y) \| = 1$ then $\| P_X(x, y) \| = \| x \| \leq 1$.

For $(x, 0)$ with $\| (x, 0) \| = \| x \| = 1$, we see that $\| P_X \| = 1$ and similarly $\| P_Y \| = 1$. So $P_X$ and $P_Y$ are bounded.

Since $X$ and $Y$ are Banach spaces, then $X \times Y$ is a Banach space (see pages 51-52 and the “claim” on page 1 of the class notes for Section 2.10).

Since $G_T$ is closed in $X \times Y$, then $G_T$ is a Banach space by Theorem 2.16.

The range of $P_X$ is all of $X$ since $T$ is defined on $X$, and so $R(P_X) = X$ is closed. So by Theorem 3.6, $P^{-1}_X$ is bounded.

So $T = P_Y P^{-1}_X$ is bounded by Proposition 2.8. (Notice $P^{-1}_X: X \rightarrow G_T$ and $P_Y: G_T \rightarrow Y$, so $T = P_Y P^{-1}_X$.)

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Suppose \( T \) is closed. Let \( P_X \) be the projection \( P_X : G_T \to X \) defined as \( P_X(x, y) = x \), and let \( P_Y : G_T \to Y \) be defined as \( P_Y(x, y) = y \).
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