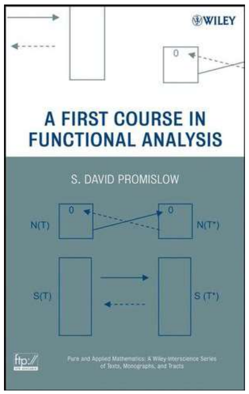
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems

3.6. Uniform Boundedness Principle—Proofs of Theorems



Theorem 3.10. Uniform Boundedness Principle

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If X is complete, then a pointwise bounded subset $\mathcal{A}$ of $\mathcal{B}(X,Y)$ is bounded.

Proof. We replace Y with its completion using Theorem 2.22. We now show boundedness on the completion of Y, which certainly implies boundedness on Y itself. Let $\mathcal{A} \subseteq \mathcal{B}(X,Y)$ be a pointwise bounded set. Define the direct product $\mathcal{Y}$ over set $\mathcal{A}$ with each space equal to Y: $\mathcal{Y} = \prod_{T\in\mathcal{A}} Y$. Then the elements of the direct product are the bounded functions (under the sup norm) mapping $\mathcal{A}$ to Y (see the definition of direct product in Section 2.10. Direct Products and Sums).

Define $\mathcal{T}x(T) = Tx$ for all $T \in \mathcal{A}$. So $\mathcal{T}$ has input value x and produces $\mathcal{T}x \in \mathcal{Y}$, where $\mathcal{T}x \in \mathcal{Y}$ is a mapping of $\mathcal{A}$ into Y:

$$\mathcal{T} : X \to \mathcal{Y} \text{ and for } x \in X \text{ we have } \mathcal{T}x : \mathcal{A} \to Y.$$

Theorem 3.10 (continued 1)

Proof (continued). Since $\mathcal{A}$ is hypothesized to be pointwise bounded, for each $x \in X$, $\mathcal{O}_x = \{\|Tx\| \mid T \in \mathcal{A}\}$ is bounded, say by K_x. So for $\mathcal{T}x \in \mathcal{Y}$,

$$\|\mathcal{T}x\| = \sup\{\|\mathcal{T}xT\| \mid T \in \mathcal{A}, \|T\| = 1\}$$
$$= \sup\{\|Tx\| \mid T \in \mathcal{A}, \|T\| = 1\} \le K_x.$$

So $\mathcal{T}x$ is a bounded function on $\mathcal{A}$ for each $x \in X$.

Now for the boundedness of $\mathcal{A}$. Suppose that $(x_n) \subseteq X$ converges to $x \in X$, and suppose that $(\mathcal{T}x_n) \subseteq \mathcal{A}$ converges to some $g : \mathcal{A} \to Y$. We now show that $g = \mathcal{T}x$ (by showing they agree for all $T \in \mathcal{A}$). Indeed,

$$\begin{aligned} g(T) &= \lim_{n\to\infty} (\mathcal{T}x_n)(T) \text{ by assumption on convergence of } (\mathcal{T}x_n) \\ &= \lim_{n\to\infty} Tx_n \text{ since } \mathcal{T}x_n(T) = Tx_n \text{ by the definition of } \mathcal{T} \\ &= T(\lim x_n) \text{ since } T \in \mathcal{A} \subseteq \mathcal{B}(X,Y) \text{ is bounded and so} \\ &\quad\ \text{continuous by Theorem 2.6} \\ &= Tx \text{ since } (x_n) \to x \ldots \end{aligned}$$

Theorem 3.10 (continued 2)

Proof (continued). ...

$$\begin{aligned} g(T) &= Tx \text{ since } (x_n) \to x \\ &= \mathcal{T}x(T) \text{ by definition of } \mathcal{T}. \end{aligned}$$

So for any sequence $(x_n, \mathcal{T}x_n) \to (x, \mathcal{T}x)$ in $X \times \mathcal{Y}$ (with respect to the sup norm on $X \times \mathcal{Y}$), we have $(x_n) \to x$ and $(\mathcal{T}x_n) \to \mathcal{T}x$. Hence the graph of $\mathcal{T}$ is closed. So by the Closed Graph Theorem (Theorem 3.9), $\mathcal{T}$ is bounded. That is,

$$\|\mathcal{T}\| = \sup\{\|\mathcal{T}x\| \mid x \in X, \|x\| = 1\} = K < \infty.$$

Hence, for any $T \in \mathcal{A}$ and for any $x \in X$ with $\|x\| = 1$, we have $\|\mathcal{T}x(T)\| = \|Tx\| \le K$. Since $\|Tx\| \le K$ for all unit vectors $x \in X$, then $\|T\| \le K$. This holds for all $T \in \mathcal{A}$ so that $\mathcal{A}$ is bounded (by K), as claimed. $\square$

Theorem 3.11

Theorem 3.11. Suppose that (T_n) is a pointwise convergent sequence of bounded linear operators from Banach space X to normed linear space Y. That is, for each $x \in X$ the sequence $(T_n x)$ converges to an element $Tx \in Y$. Then T is linear and bounded.

Proof. For linearity, let $\alpha x + \beta y \in X$. Then

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y))$$

$$= \alpha \lim_{n\to\infty} (T_n(x)) + \beta \lim_{n\to\infty} (T_n(x)) = \alpha T(x) + \beta T(y)$$

and T is linear, as claimed.

Now for boundedness. For $x \in X$, the sequence $(T_n x)$ converges and so is bounded by Theorem 2.9(a) and (b). So $\{T_n \mid n \in \mathbb{N}\}$ is a pointwise bounded set; that is, for each $x \in X$ the set $\mathcal{O}_x = \{T_n x \mid n \in \mathbb{N}\}$ is bounded. By the Uniform Boundedness Principle, there is $K > 0$ such that $\|T_n\| \leq K$ for all $n \in \mathbb{N}$.

Theorem 3.11 (continued)

Theorem 3.11. Suppose that (T_n) is a pointwise convergent sequence of bounded linear operators from Banach space X to normed linear space Y. That is, for each $x \in X$ the sequence $(T_n x)$ converges to an element $Tx \in Y$. Then T is linear and bounded.

Proof (continued). Then, for $x \in X$ a unit vector, we have

$$\begin{aligned} \|Tx\| &= \|\lim T_n x\| \text{ by the convergence hypothesis} \\ &= \lim \|T_n x\| \text{ since } \|\cdot\| \text{ is continuous by Theorem 2.3(c)} \\ &\leq K. \end{aligned}$$

So $\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\} \leq K$ and T is bounded, as claimed. $\square$