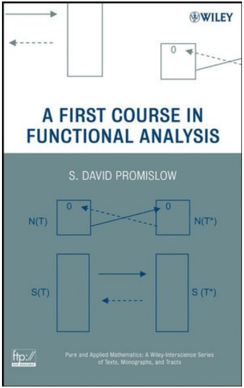
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.2. Semi-Inner Products—Proofs of Theorems



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Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\text{Re}\langle x, y\rangle.$$

Proof. We have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x + y\rangle + \langle y, x + y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\text{Re}(\langle x, y\rangle) + \|y\|^2. \end{aligned}$$

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Theorem 4.3. Cauchy-Schwartz Inequality

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Given a semi-inner product on X, for all $x, y \in X$ we have $|\langle x, y\rangle| \le \|x\|\|y\|$. If $\langle \cdot, \cdot \rangle$ is an inner product, then equality holds if and only if x is a scalar multiple of y (that is, x and y are linearly dependent).

Proof. Notice that if we scale x or y by a nonzero amount, the Cauchy-Schwartz Inequality remains unchanged (the scaling factor is introduced on both sides and can be divided out).

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Suppose $\|y\| \neq 0$. If $\langle x, y\rangle = 0$, the result trivially follows. If $\langle x, y\rangle \neq 0$ then x and y can be scaled so that $\|y\| = 1$ and $\langle x, y\rangle = 1$ (the second claim possibly requiring the use of a complex factor). Then by Lemma 4.2, with y replaced by $-y$, we have

$$0 \leq \|x-y\|^2 = \|x\|^2+\|-y\|^2+2\text{Re}(\langle x, -y\rangle) = \|x\|^2+1+2(-1) = \|x\|^2-1.$$

So $\|x\| \geq 1$ and the result holds in the case that $\|y\| \neq 0$ (or symmetrically, in the case that $\|x\| \neq 0$).

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Proof (continued). If $\|x\| = \|y\| = 0$ but $\langle x, y\rangle \neq 0$, then we can scale x and y such that $\langle x, y\rangle = 1$. Then by Lemma 4.2,

$$0 \leq \|x - y\|^2 = \|x\|^2 + \|-y\|^2 + 2\mathrm{Re}(\langle x, -y\rangle) = 0 + 0 - 2,$$

a contradiction (so it must be that $\langle x, y\rangle = 0$) and the inequality follows. If $y = \alpha x$ then

$$|\langle x, y\rangle| = |\langle x, \alpha x\rangle| = |\overline{\alpha}\langle x, x\rangle| = |\alpha|\|x\|^2 = \|x\||\alpha|\|x\| = \|x\|\|y\|.$$

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Proof (continued). Conversely, suppose that equality holds. Notice that $\langle 0, x\rangle = \langle x - x, x\rangle = \langle x, x\rangle - \langle x, x\rangle = 0$, so if either $\|x\| = 0$ or $\|y\| = 0$, then x and y are linearly dependent since $x = 0$ or $y = 0$ (we are assuming $\langle\cdot,\cdot\rangle$ is an inner product and $\|\cdot\|$ is a norm in the exploration of equality). If neither x nor y is 0, then again we can scale y so that $\|y\| = 1$ and then scale x so that $\langle x, y\rangle = 1$. Then equality in the Cauchy-Schwartz Inequality implies that $\|x\| = 1$. Then, as above, with $-y$ replacing y in Lemma 4.2,

$$0 \leq \|x - y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, -y\rangle) = 1 + 1 + 2(-1) = 0,$$

and so $\|x - y\| = 0$ and $x = y$ □

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Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have $\|x+y\| \leq \|x\| + \|y\|$.

Proof. We simply have

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, y\rangle) \text{ by Lemma 4.2} \\ &\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \text{ by Cauchy-Schwartz} \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Taking square roots, we have

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Theorem 4.5. Parallelogram Law

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Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. Lemma 4.2 gives $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\text{Re}(\langle x, y\rangle)$ and, by replacing y with $-y$, gives

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 + 2\text{Re}(\langle x, -y\rangle) = \|x\|^2 + \|y\|^2 - 2\text{Re}(\langle x, y\rangle).$$

Adding these two equations give the result. □

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Proposition 4.7. Polarization Identity

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Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right).$$

Proof. As in the proof of the Parallelogram Law
$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\text{Re}(\langle x, y \rangle)$ and
$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\text{Re}(\langle x, y \rangle)$. Subtracting these give

$$4\text{Re}(\langle x, y \rangle) = \|x + y\|^2 - \|x - y\|^2. \; (*)$$

As with any complex number, $\langle x, y \rangle = \text{Re}(\langle x, y \rangle) + i\text{Im}(\langle x, y \rangle)$ and so

$$\text{Im}(\langle x, y \rangle) = \text{Re}(-i\langle x, y \rangle) = \text{Re}(\langle x, iy \rangle). \; (**)$$

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Proof (continued). Using $(*)$ for $\mathrm{Re}(\langle x, y \rangle)$ and using $(*)$ with y replaced by iy gives

$$\mathrm{Re}(\langle x, y \rangle) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

and

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and this substituted into $(**)$ gives the result. □

Theorem 4.9. Continuity of Inner Product

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In any semi-inner product space, if the sequences $(x_n) \to x$ and $(y_n) \to y$, then $(\langle x_n, y_n \rangle) \to \langle x, y \rangle$.

Proof. We have

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \\ &\leq |\langle x_n, y_n \rangle - \langle x_n, y \rangle| + |\langle x_n, y \rangle - \langle x, y \rangle| \\ &\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \\ &\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \text{ by Cauchy-Schwartz.} \end{aligned}$$

Since (x_n) is convergent, then it is bounded, since $(y_n) \to y$ then $\|y_n - y\| \to 0$, since $(x_n) \to x$ then $\|x_n - x\| \to 0$. Therefore

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| \to 0 \text{ and } \langle x_n, y_n \rangle \to \langle x, y \rangle.$$

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