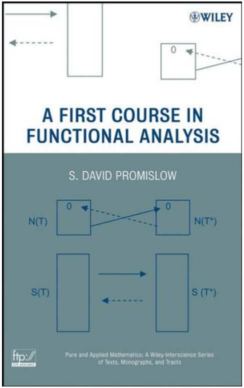
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.3. Nearest Points and Convexity—Proofs of Theorems



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Proof. We translate point x and set K by an amount $-x$, so that x goes to 0 and K goes to the set $K - x$. Then finding a point in K nearest to x is equivalent to finding a point in $K - x$ of minimal norm. ASSUME there are two points $y, z \in K - x$ of minimal norm, say $\|y\| = \|z\| = a$. Then, since X is strictly convex, $\|(y/a + z/a)/2\| < 1$, or $\|(y + z)/2\| < a$. But then $(y + z)/2 \in K - x$ since $K - x$ is convex, and $(y + z)/2$ is of smaller norm than y and z, CONTRADICTING the minimality of the norm of y and z. So the assumption of two points of minimal norm is false and there is at most on point in $K - x$ of minimal norm, and hence at most one point in K that is nearest to x. □

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Lemma 4.3.A

Lemma 4.3.A. If a normed linear space is uniformly convex, then it is strictly convex.

Proof. We consider the contrapositive. Suppose a linear space is not strictly convex. Then there are distinct unit vectors x and y such that $\|(x+y)/2\| = 1$. Let $\varepsilon = \|x-y\|$. Then for all $\delta > 0$ we have $\left\|\frac{1}{2}(x+y)\right\| = 1 > 1-\delta$, but we do not have $\|x-y\| = \varepsilon < \varepsilon$. Therefore, X is not uniformly convex. $\square$

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Theorem 4.12

Theorem 4.12. Suppose X is a uniformly convex Banach space. For any point x and a nonempty closed convex set K, there is a nearest point to x in K.

Proof. By translating by $-x$ and then scaling by $d(x, K)$, we can reduce the existence problem to showing that $d(0, K) = 1$ implies there is a point $y \in K$ where $\|y\| = 1$. Now $d(0, K) = \inf\{\|y\| \mid y \in K\} = 1$, then for each $n \in \mathbb{N}$ there is $y_n \in K$ with $1 \leq \|y_n\| \leq 1 + 1/n$.

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$$\begin{aligned}\|(y_m+y_n)/2\| &\ge 1 \text{ since } d(0,K)=1 \text{ and } (y_m+y_n)/2\in K\\ &= \frac{N+1}{N}\left(1-\frac{1}{N+1}\right)\\ &> \frac{N+1}{N}(1-\delta) \text{ since } \frac{1}{N+1}<\frac{1}{N}<\delta. \qquad (*)\end{aligned}$$

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$$\begin{aligned}
\|(y_m + y_n)/2\| &\geq 1 \text{ since } d(0, K) = 1 \text{ and } (y_m + y_n)/2 \in K \\
&= \frac{N+1}{N}\left(1 - \frac{1}{N+1}\right) \\
&> \frac{N+1}{N}(1 - \delta) \text{ since } \frac{1}{N+1} < \frac{1}{N} < \delta. \qquad (*)
\end{aligned}$$

Theorem 4.12 (continued 1)

Proof (continued). Define $\hat{y}_n = \frac{N}{N+1} y_n$ and $\hat{y}_m = \frac{N}{N+1} y_m$. Then

$$\|\hat{y}_n\| = \frac{N}{N+1}\|y_n\| \leq \frac{N}{N+1}\left(1 + \frac{1}{n}\right) = \frac{N}{N+1}\frac{n+1}{n} \leq 1$$

since $f(x) = 1 + 1/x$ is decreasing and $n \geq N$. Similarly $\|\hat{y}_m\| \leq 1$ and so $\hat{y}_n, \hat{y}_m \in \overline{B}(1)$ and

$$\begin{aligned} \left\|\frac{\hat{y}_n + \hat{y}_m}{2}\right\| &= \left\|\left(\frac{N}{N+1}y_n + \frac{N}{N+1}y_m\right)\Big/ 2\right\| = \frac{N}{N+1}\left\|\frac{y_n + y_m}{2}\right\| \\ &> \frac{N}{N+1}\frac{N+1}{N}(1-\delta) \text{ by } (*) \\ &= 1 - \delta. \end{aligned}$$

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&> \frac{N}{N+1}\frac{N+1}{N}(1-\delta) \text{ by } (*) \\
&= 1 - \delta.
\end{aligned}$$

Theorem 4.12 (continued 2)

Theorem 4.12. Suppose X is a uniformly convex Banach space. For any point x and a nonempty closed convex set K, there is a nearest point to x in K.

Proof (continued). So, by the uniform convexity hypothesis and the choice of δ, $\|\hat{y}_n - \hat{y}_m\| < \varepsilon/2$. Hence

$$\|y_m - y_n\| = \left\| \frac{N+1}{N}\hat{y}_m - \frac{N+1}{N}\hat{y}_n \right\| = \frac{N+1}{N}\|\hat{y}_m - \hat{y}_n\| < \frac{N+1}{N}\frac{\varepsilon}{2} \leq \varepsilon.$$

So (y_n) is a Cauchy sequence of elements of K. Since X is a Banach space, $(y_n) \to y$ for some $y \in X$ and since K is closed, $y \in K$. Also since $1 \leq \|y_n\| \leq 1 + 1/n$ for each n, then $\|y\| = 1$ by Continuity of the Norm (Theorem 2.3(c)). So $y \in K$ has norm 1 and the general result follows, as explained above. □