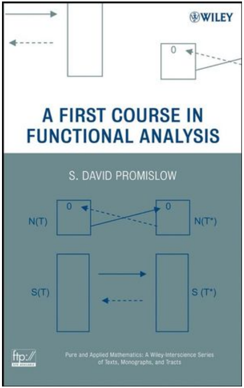
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.4. Orthogonality—Proofs of Theorems



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Proposition 4.13

Proposition 4.13. Suppose S is a subset of a Hilbert space H and suppose S is closed under scalar multiplication (i.e., $y \in S$ and $\alpha \in \mathbb{C}$ implies $\alpha y \in S$). Then

$$S^{\perp} = \{x \in H \mid d(x, S) = \|x\|\}.$$

Proof. If $d(x, S) = \|x\|$, then for all $y \in S$ and $t > 0$,

$$\begin{aligned} \|x\|^2 &= (\inf\{\|x - y\| \mid y \in S\})^2 \leq \|x \pm ty\|^2 \text{ since } ty \in S \\ &= \|x\|^2 + t^2\|y\|^2 \pm 2\mathrm{Re}(t\langle x, y\rangle) \text{ by Lemma 4.2.} \end{aligned}$$

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So $\mp\mathrm{Re}(\langle x, y\rangle) \le (t/2)\|y\|^2$ holds for all $t > 0$ and so it must be that $\mathrm{Re}(\langle x, y\rangle) = 0$. Similarly, by considering that $\|x\|^2 \le \|x \pm ity\|^2$, we have that $\mathrm{Im}(\langle x, y\rangle) = \mathrm{Re}(\langle x, iy\rangle) = 0$. So $\langle x, y\rangle = 0$ and $x \in S^\perp$. Hence $\{x \in H \mid d(x, S) = \|x\|\} \subseteq S^\perp$.

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Proposition 4.13 (continued)

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$$S^\perp = \{x \in H \mid d(x, S) = \|x\|\}.$$

Proof (continued). Conversely, if $x \in S^\perp$ then for all $y \in S$,

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\text{Re}(\langle x, y\rangle) = \|x\|^2 + \|y\|^2 \geq \|x\|^2.$$

So $d(x, S) \geq \|x\|$. Since $0 \in S$ (S is closed under scalar multiplication) then

$$\|x\| \leq d(x, S) = \inf\{\|x - y\| \mid y \in S\} \leq \|x - 0\| = \|x\|$$

and $d(x, S) = \|x\|$, so $x \in \{x \in H \mid d(x, S) = \|x\|\}$, and $S^\perp \subseteq \{x \in H \mid d(x, S) = \|x\|\}$. $\square$

Theorem 4.14

Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

(a) $M \cap M^{\perp} = \{0\}$.

(b) Any $z \in H$ can be written uniquely as the sum of an element of M and an element of $M^{\perp}$. Specifically, $z = P_M(z) + P_{M^{\perp}}(z)$.

(c) $M^{\perp\perp} = M$.

(d) H is isometric to $M \oplus M^{\perp}$ where the direct sum is equipped with the ℓ^2 norm.

Proof of (a). If $x \in M \cap M^{\perp}$ then $\langle x, x \rangle = 0$ and so $x = 0$ and (a) follows.

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Proof (continued). (b). For $z \in H$, define $y = z - P_M(z)$. Now $P_M(z)$ is the vector in M nearest z. That is, $\inf\{\|z - m\| \mid m \in M\} = \|z - P_M(z)\|$. For any $m \in M$, we have

$$\|y - m\| = \|(z - P_M(z)) - m\| = \|z - (P_M(z) + m)\|$$

$$\geq \inf\{\|z - m\| \mid m \in M\} = \|z - P_M(z)\| = \|y\|.$$

So $d(y, M) = \inf\{\|y - m\| \mid m \in M\} \geq \|y\|$ and since $0 \in M$, $d(y, M) = \|y\|$. So by Proposition 4.13, $y = z - P_M(z) \in M^\perp$.

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So $d(y, M) = \inf\{\|y - m\| \mid m \in M\} \geq \|y\|$ and since $0 \in M$, $d(y, M) = \|y\|$. So by Proposition 4.13, $y = z - P_M(z) \in M^\perp$.

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Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

(b) Any $z \in H$ can be written uniquely as the sum of an element of M and an element of $M^\perp$. Specifically, $z = P_M(z) + P_{M^\perp}(z)$.

(c) $M^{\perp\perp} = M$.

Proof (continued). Now we write $z = P_M(z) + (z - P_M(z))$ where $P_M(z) \in M$ and $y = z - P_M(z) \in M^\perp$. So we see that any $z \in H$ can be written as a sum of an element of M and an element of $M^\perp$. We'll see below that this representation is unique.

Theorem 4.14 (continued 3)

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Let M be a closed subspace of a Hilbert space H. Then:

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Proof (continued). (c). $M^\perp$ is the set of all vectors in H which are orthogonal to M, and $M^{\perp\perp}$ is the set of all vectors orthogonal to $M^\perp$ (which includes all vectors in M). So $M \subseteq M^{\perp\perp}$. Now, suppose $z \in M^{\perp\perp} \subseteq H$. Then, as argued above, $z = x + y$ for some $x \in M$ and $y \in M^\perp$. Since $M \subseteq M^{\perp\perp}$, then $x \in M^{\perp\perp}$ and so $y = z - x \in M^{\perp\perp}$ ("clearly" a perp space is closed under linear combinations). So $y \in M^\perp$ and $y \in M^{\perp\perp}$ and by (a), $y = z - x = 0$ and $z = x \in M$. That is, $M^{\perp\perp} \subseteq M$. Therefore, $M = M^{\perp\perp}$ and (c) follows.

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Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

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Theorem 4.14 (continued 4)

Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

(b) Any $z \in H$ can be written uniquely as the sum of an element of M and an element of $M^\perp$. Specifically, $z = P_M(z) + P_{M^\perp}(z)$.

Proof (continued). (b) (continued). To complete (b), we now show uniqueness. Suppose $z = x_1 + y_1 = x_2 + y_2$ where $x_1, x_2 \in M$ and $y_1, y_2 \in M^\perp$. But then by (a), $x_1 - x_2 = y_2 - y_1 = 0$ and so $x_1 = x_2$ and $y_1 = y_2$ and the representation of z as a sum of an element of M and an element of $M^\perp$ is unique. Now, following the technique above but by replacing closed subspace M with closed subspace $M^\perp$, we have $z = P_{M^\perp}(z) + (z - P_{M^\perp}(z))$ where $P_{M^\perp}(z) \in M^\perp$ and $z - P_{M^\perp}(z) \in M^{\perp\perp} = M$ by (c). By the uniqueness representation of z we have $z = P_M(z) + P_{M^\perp}(z)$ and (b) follows.

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Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

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Theorem 4.14 (continued 5)

Theorem 4.14. Projection Theorem.
Let M be a closed subspace of a Hilbert space H. Then:

(d) H is isometric to $M \oplus M^\perp$ where the direct sum is equipped with the ℓ^2 norm.

Proof (continued). (d). Define $\pi : H \to M \oplus M^\perp$ as $\pi(z) = (P_M(z), P_{M^\perp}(z))$. Then "clearly" π is one to one, onto, and preserves linear combinations (i.e., π is a linear space isomorphism). Now for $z \in H$,

$$\begin{aligned} \|\pi(z)\|_2^2 &= \|(P_M(z), P_{M^\perp}(z))\|_2^2 \\ &= \|P_M(z)\|^2 + \|P_{M^\perp}(z)\|^2 \text{ since we are using the } \ell^2 \text{ norm} \\ &= \|z\|^2 \text{ by Lemma 4.2 since } \langle P_M(z), P_{M^\perp}(z)\rangle = 0. \end{aligned}$$

So π is an isometry, and (d) follows. □

Theorem 4.18

Theorem 4.18. Every Hilbert space has an orthonormal basis.

Proof. Let P be the class whose members are all orthonormal subsets of the Hilbert space H. Define the partial order $\leq$ on P as $A \leq B$ for $A, B \in P$ if $A \subset B$. Now for any nonzero $x \in H$, we have $\{x/\|x\|\} \in P$, so P is nonempty.

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