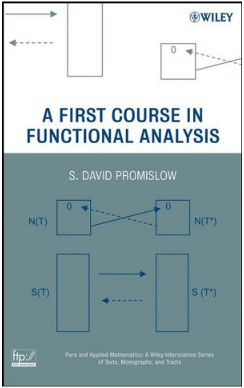
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.6. Linear Operators on Hilbert Spaces—Proofs of Theorems



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Theorem 4.24

Theorem 4.24. Given any $T \in \mathcal{B}(H)$ (the set of bounded linear transformations from H to itself), the function f_T defined by $f_T(x, y) = \langle Tx, y\rangle$ is a sesquilinear form with norm equal to $\|T\|$. Conversely, given any bounded sesquilinear form f, there is a unique $T \in \mathcal{B}(H)$ such that $f = f_T$.

Proof. Since T is linear and $\langle \cdot, \cdot\rangle$ is sesquilinear, then f_T is sesquilinear.

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Proof. Since T is linear and $\langle\cdot,\cdot\rangle$ is sesquilinear, then f_T is sesquilinear. By the Cauchy-Schwarz Inequality (Theorem 4.3) we have

$$\begin{aligned} \|f_T\| &= \sup \||f(x,y)| \mid \|x = \|y\| = 1\} \\ &= \sup\{|\langle Tx, y\rangle \mid \|x\| = \|y\| = 1\} \le \|Tx\|\|y\| \\ &\le \|T\|\|x\|y\| = \|T\|. \end{aligned}$$

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Clearly $\|f_T\| = \|T\|$ if $T = 0$. If $T \neq 0$ then $Tx \neq 0$ for some unit vector x. Let $y = Tx/\|Tx\|$. Then $f_T(x, y) = \langle Tx, Tx/\|Tx\|\rangle = \|Tx\|$, and so $\|f_T\| \geq \|T\|$. Therefore $\|f_T\| = \|T\|$.

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Theorem 4.24 (Part 2)

Proof (continued). Conversely, suppose f is a bounded sesquilinear form. Fix $x \in H$ and define $g : H \to \mathbb{C}$ as $g(y) = \overline{f(x, y)}$.

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$$\begin{aligned} |g(y)| &= |\overline{f(x,y)}| = |f(x,y)| \\ &= |\|x\| f(x/\|x\|, y)| \text{ by linearity in the first position} \\ &\leq \|x\| \|f\|, \end{aligned}$$

so g is a bounded linear functional on H.

Theorem 4.24 (Part 2)

Proof (continued). Conversely, suppose f is a bounded sesquilinear form. Fix $x \in H$ and define $g : H \to \mathbb{C}$ as $g(y) = \overline{f(x,y)}$. Then for $\|y\| = 1$,

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so g is a bounded linear functional on H. By Theorem 4.22, there is a unique $x \in H$ such that $g = \psi_z$ (i.e., $g(y) = \langle y, z\rangle = \psi_z$). Define $T : H \to H$ on the fixed x as $Tx = z$.

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Proof (continued). Conversely, suppose f is a bounded sesquilinear form. Fix $x \in H$ and define $g : H \to \mathbb{C}$ as $g(y) = \overline{f(x,y)}$. Then for $\|y\| = 1$,

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so g is a bounded linear functional on H. By Theorem 4.22, there is a unique $x \in H$ such that $g = \psi_z$ (i.e., $g(y) = \langle y, z\rangle = \psi_z$). Define $T : H \to H$ on the fixed x as $Tx = z$. Now consider $\alpha x_1 + \beta x_2$. We have for some $z_3 \in H$, $T(\alpha x_1 + \beta x_2) = z_3$ where $g_{z_3}(y) = \overline{f(\alpha x_1 + \beta x_2, y)} = \psi_{z_3}$.

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Theorem 4.24 (Part 3)

Proof (continued). Next

$$
\begin{aligned}
f(\alpha x_1 + \beta x_2, y) &= \overline{g_{z_3}(y)} = \overline{\psi_{z_3}(y)} \\
&= \overline{\langle y, z_3 \rangle} = \langle z_3, y \rangle = \langle T(\alpha x_1 + \beta x_2), y \rangle \\
&= \alpha f(x_1, y) + \beta f(x_2, y) \text{ since } f \text{ is sesquilinear} \\
&= \alpha \langle T x_1, y \rangle + \beta \langle T x_2, y \rangle \\
&= \langle \alpha T x_1 + \beta T x_2, y \rangle \text{ since } \langle \cdot, \cdot \rangle \text{ is linear in first position.}
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Proof (continued). Next

$$\begin{aligned} f(\alpha x_1+\beta x_2,y) &= \overline{g_{z_3}(y)} = \overline{\psi_{z_3}(y)} \\ &= \overline{\langle y,z_3\rangle} = \langle z_3,y\rangle = \langle T(\alpha x_1+\beta x_2),y\rangle \\ &= \alpha f(x_1,y)+\beta f(x_2,y) \text{ since } f \text{ is sesquilinear} \\ &= \alpha\langle Tx_1,y\rangle+\beta\langle Tx_2,y\rangle \\ &= \langle \alpha Tx_1+\beta Tx_2,y\rangle \text{ since } \langle\cdot,\cdot\rangle \text{ is linear in first position.} \end{aligned}$$

Therefore, $\langle T(\alpha x_1+\beta x_2),y\rangle = \langle \alpha Tx_1+\beta Tx_2,y\rangle$ for all y and by linearity in the first position $\langle T(\alpha x_1+\beta x_2)-(\alpha Tx_1+\beta Tx_2),y\rangle = 0$ for all $y\in H$. So, $T(\alpha x_1+\beta x_2)=\alpha Tx_1+\beta Tx_2$ and T is linear.

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Moreover, for all $y\in H$,

$$f_T(x,y)=\langle Tx,y\rangle=\langle z,y\rangle\overline{\langle y,z\rangle}=\overline{\psi_z(y)}=\overline{g(y)}=f(x,y).$$

As argued above, $\|f\|=\|f_T\|=\|T\|$ and so T is bounded.

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Proof (continued). Next

$$\begin{aligned} f(\alpha x_1 + \beta x_2, y) &= \overline{g_{z_3}(y)} = \overline{\psi_{z_3}(y)} \\ &= \overline{\langle y, z_3\rangle} = \langle z_3, y\rangle = \langle T(\alpha x_1 + \beta x_2), y\rangle \\ &= \alpha f(x_1, y) + \beta f(x_2, y) \text{ since } f \text{ is sesquilinear} \\ &= \alpha\langle Tx_1, y\rangle + \beta\langle Tx_2, y\rangle \\ &= \langle \alpha Tx_1 + \beta Tx_2, y\rangle \text{ since } \langle\cdot,\cdot\rangle \text{ is linear in first position.} \end{aligned}$$

Therefore, $\langle T(\alpha x_1 + \beta x_2), y\rangle = \langle \alpha Tx_1 + \beta Tx_2, y\rangle$ for all y and by linearity in the first position $\langle T(\alpha x_1 + \beta x_2) - (\alpha Tx_1 + \beta Tx_2), y\rangle = 0$ for all $y \in H$. So, $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$ and T is linear.
Moreover, for all $y \in H$,

$$f_T(x, y) = \langle Tx, y\rangle = \langle z, y\rangle \overline{\langle y, z\rangle} = \overline{\psi_z(y)} = \overline{g(y)} = f(x, y).$$

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Proof (continued). If there were two such T, say T and T', where $f_T = f_{T'}$, then for all $x, y \in H$ we would have $\langle Tx, y \rangle = \langle Tx, y \rangle$, or $\langle Tx - T'x, y \rangle = 0$ for all $x, y \in H$ and hence $Tx - T'x = 0$, or $Tx = T'x$ for all $x \in H$. That is, $T = T'$ and uniqueness follows. $\square$

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Theorem 4.26, Properties of Hilbert Space Adjoints

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Given $S, Y \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

(a) $(S + T)^* = S^* + T^*$

(b) $(\alpha T)^* = \overline{\alpha} T^*$

(c) $(ST)^* = T^* S^*$

(d) $\|T^*\| = \|T\|$

(e) $T^{**} = T$

(f) $\|T^* T\| = \|T\|^2$.

Proof of (a). For all $x, y \in H$, we have $\langle (S + T)x, y\rangle = \langle x, (S + T)^*, y\rangle$. Then

$$\langle (S+T)x, y\rangle = \langle Sx, y\rangle + \langle Tx, y\rangle = \langle x, S^* y\rangle + \langle x, T^* y\rangle = \langle x, (S^* + T^*)y\rangle.$$

So $(S + T)^* = S^* + T^*$. □

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Proof of (a). For all $x, y \in H$, we have $\langle (S+T)x, y\rangle = \langle x, (S+T)^*, y\rangle$. Then

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Theorem 4.26(b) and (c)

Theorem 4.26. Properties of Hilbert Space Adjoints.
Given $S, Y \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

(b) $(\alpha T)^* = \overline{\alpha} T^*$

(c) $(ST)^* = T^* S^*$

Proof of (b). For all $x, y \in H$, we have $\langle (\alpha T)x, y \rangle = \langle x, (\alpha T)^* y \rangle$. Then

$$\langle (\alpha T)x, y \rangle = \alpha \langle Tx, y \rangle = \alpha \langle x, T^* y \rangle = \langle x, \overline{\alpha} T^* y \rangle.$$

So $\alpha T)^* = \overline{\alpha} T^*$. □

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Proof of (c). For all $x, y \in H$, we have

$$\langle (ST)^* x, y \rangle = \langle x, (ST)y \rangle = \langle x, S(Ty) \rangle = \langle S^* x, Ty \rangle = \langle T^* S^* x, y \rangle$$

and so $(ST)^* = T^* S^*$. □

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Proof of (c). For all $x, y \in H$, we have

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Theorem 4.26(d)

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Given $S, Y \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

(d) $\|T^*\| = \|T\|$

Proof of (d). For $f_T = \langle Tx, y\rangle = f_{T^*} = \langle x, T^*y\rangle$, we have

$$\begin{aligned} \|f_T\| &= \sup\{|f_T(x,y)| \mid \|x\| = \|y\| = 1\} \\ &= \sup\{|\langle Tx, y\rangle| \mid \|x\| = \|y\| = 1\} \\ &= \sup\{|\langle x, Ty\rangle| | \|x\| = \|y\| = 1\} \text{ since } \langle Tx, y\rangle = \overline{\langle y, Tx\rangle} \text{ and the} \\ &\qquad \text{sup is taken over all } \|x\| = \|y\| = 1 \\ &= \|T\| \text{ by Theorem 4.24.} \end{aligned}$$

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Similarly, $\|f_{T^*}\| = \|T^*\|$ and since $f_T = f_{T^*}$, then $\|T\| = \|T^*\|$. □

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Theorem 4.26(e) and (f)

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Given $S, Y \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

(e) $T^{**} = T$

(f) $\|T^*T\| = \|T\|^2$.

Proof of (e). For all $x, y \in H$ we have $\langle T^{**}x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle$, so $T^{**} = T$. □

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Proof of (f). By Proposition 2.8, $\|T^* T\| \leq \|T^*\| \|T\| = \|T\|^2$ by part (e).

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Proof of (f). By Proposition 2.8, $\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2$ by part (e). If x is a unit vector then

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \\ &\le \|T^*T\| \text{ by definition of the operator norm.} \end{aligned}$$

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Proposition 4.27

Proposition 4.27. For all $T \in \mathcal{B}(H)$ (the set of bounded linear transformations from H to itself):

(a) $N(T^*) = R(T)^\perp$

(b) $N(T)^\perp = \overline{R(T^*)}$.

Proof of (a). We have $x \in R(T)^\perp$ if and only if for all $y \in H$ we have $\langle x, Ty \rangle = 0$ (since $Ty \in R(T)$). Equivalently, $\langle T^*x, y \rangle = 0$ for all $y \in H$, which means $T^*x = 0$ and so $x \in N(T^*)$. $\square$

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Proof of (b). From (a) with T^* replacing T we have $N(T) = R(T^*)^\perp$ (since $T^{**} = T$ by Theorem 4.26(e)), and so $N(T)^\perp = R(T^*)^{\perp\perp} = \overline{R(T^*)}$ by Proposition 4.15. □

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Proposition 4.27. For all $T \in \mathcal{B}(H)$ (the set of bounded linear transformations from H to itself):

(a) $N(T^*) = R(T)^\perp$
(b) $N(T)^\perp = \overline{R(T^*)}$.

Proof of (a). We have $x \in R(T)^\perp$ if and only if for all $y \in H$ we have $\langle x, Ty \rangle = 0$ (since $Ty \in R(T)$). Equivalently, $\langle T^*x, y \rangle = 0$ for all $y \in H$, which means $T^*x = 0$ and so $x \in N(T^*)$. □

Proof of (b). From (a) with T^* replacing T we have $N(T) = R(T^*)^\perp$ (since $T^{**} = T$ by Theorem 4.26(e)), and so $N(T)^\perp = R(T^*)^{\perp\perp} = \overline{R(T^*)}$ by Proposition 4.15. □

Proposition 4.30

Proposition 4.30. T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Proof. For all $x \in H$, we have that $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$ and $\langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\rangle^2$.

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Proposition 4.31

Proposition 4.31. T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.

Proof. For all $x \in H$ we have $\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ and these are equal if and only if $\langle Tx, x\rangle$ is real.

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Proposition 4.33. An element $P \in \mathcal{B}(H)$ is a projection if and only if there is a closed subspace M of H such that $P = P_M$ (the projection onto M, see page 79).

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$$Px = P(y + z) = P(y) + P(z) = P(y) + 0 = P(y) = y.$$

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Proposition 4.33 (continued)

Proof (continued). Conversely, the projection P_M maps elements of M into themselves (P is the identity on M), so $P_M^2 = P_M$ on H. For $x_1, x_2 \in H$ where $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ where $y_1, y_2 \in M$ and $z_1, z_2 \in M^\perp$, we have

$$\langle P_M x_1, x_2 \rangle = \langle P_M(y_1 + z_1), y_2 + z_2 \rangle = \langle y_1, y_2 + z_2 \rangle$$

$$= \langle y_1, y_2 \rangle + \langle y_1, z_2 \rangle = \langle y_1, y_2 \rangle + 0 = \langle y_1, y_2 \rangle,$$

and

$$\langle x_1, P_M x_2 \rangle = \langle y_1 + z_1, P_M(y_2 + z_2) \rangle = \langle y_1 + z_1, y_2 \rangle$$

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So $\langle P_M x_1, x_2 \rangle = \langle x_1, P_M x_2 \rangle$ for all $x_1, x_2 \in H$ and so $P_M = P_M^*$. Therefore P_M is a projection. □

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Proposition 4.34

Proposition 4.34. An element $U \in \mathcal{B}(H)$ is unitary if and only if it is a surjective (onto) isometry.

Proof. Suppose U is unitary. Then $U^*U = \mathcal{I}$ and so

$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle = \langle \mathcal{I}x, x \rangle = \langle x, x \rangle = \|x\|^2.$$

Therefore U is an isometry. For any $y \in H$, $\mathcal{I}y = UU^*y = U(U^*y) = y$ and so U is surjective (onto).

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Proof (continued). Next, suppose U is a surjective isometry. Since U is an isometry, then (as above)

$$\begin{aligned} \|Ux\|^2 &= \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle \\ &= \|x\|^2 \text{ since } U \text{ is an isometry} \\ &= \langle x, x \rangle, \end{aligned}$$

and so $\langle U^*Ux, x \rangle = \langle x, x \rangle$ for all $x \in H$ and by Corollary 4.25, $U^*U = \mathcal{I}$. Next, if $x \in H$, then $x = Uy$ for some $y \in H$ since U is surjective (onto) and so

$$UU^*(x) = UU^*(Uy) = U(U^*Uy) = Uy = x$$

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