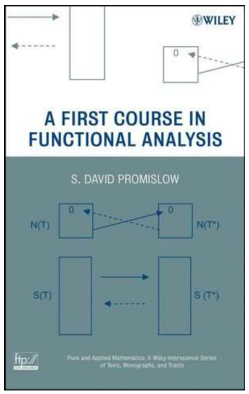
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.7. Order Relation on Self-Adjoint Operators—Proofs of Theorems



Theorem 4.38

Proposition 4.38. Given two closed subspaces M and N, the projection P_M and P_N satisfy $P_M \leq P_N$ if and only if $M \subseteq N$.

Proof. Suppose $M \subseteq N$. Then for any $x \in H$ we have

$$\begin{aligned} \langle P_M x, x \rangle &= \langle P_M P_M x, x \rangle \text{ since } P_M^2 = P_M \\ &= \langle P_M x, P_M x \rangle \text{ since } P_M^* = P_M \\ &= \|P_M x\|^2\| \\ &\leq \|P_N x\|^2 \text{ since } M \subseteq N \\ &= \langle P_N x, x \rangle \text{ (as above for } P_M). \end{aligned}$$

So $\langle P_N x, x \rangle - \langle P_M x, x \rangle \geq 0$, or $\langle (P_N - P_M)x, x, \rangle \geq 0$ for all $x \in H$, and hence $P_M \leq P_N$.

Theorem 4.38 (continued)

Proposition 4.38. Given two closed subspaces M and N, the projection P_M and P_N satisfy $P_M \leq P_N$ if and only if $M \subseteq N$.

Proof (continued). For the converse, we consider the contrapositive. Suppose $M \not\subseteq N$. Then for some unit vector $x \in M$ we have that $x \notin N$. Then $\langle P_M x, x \rangle = \langle x, x \rangle = 1$, but

$$\begin{aligned} \langle P_n x, x \rangle &= \langle P_N x, P_N x \rangle \text{ (as above)} \\ &+ \|P_N x\|^2 < 1 \text{ since } x \notin N. \end{aligned}$$

Then for this x we have that

$$\langle (P_N - P_M)x, x \rangle = \langle P_N x, x \rangle - \langle P_M x, x \rangle < 0$$

and so we do not have $P_M \leq P_N$. The result follows. □