and so we do not have $N_d \subseteq M_d$. The result follows.

$$0 > \langle x', x N_d \rangle - \langle x', x (M_d - N_d) \rangle$$

Then for this $x$ we have that

$$N \not\supseteq x_{||x||^2}$$

Since $x \in N$.

Then $\langle x, x N_d \rangle = \langle x, x' d \rangle$.

Suppose $N_d \not\subseteq M_d$. Then for some unit vector $x \in M$ we have that $x \not\in N$.

**Proposition (continued).** For the converse, we consider the contrapositive.

**Proposition 4.38.** Given two closed subspaces $M$ and $N$, the projection $P_d$ and $N_d$ satisfy $P_d \subseteq M$ if and only if $P_d \subseteq N$.

**Theorem 4.38 (continued).**

Hence $P_d \subseteq M_d$. Since $P_d$ is above for all $x \in H$, and

$$\langle x', x (P_d - N_d) \rangle = 0$$

So

$$\langle x', x N_d \rangle = \langle x', x P_d \rangle$$

Since $\langle x, x N_d \rangle = \langle x, x P_d \rangle$.

$$P_d = \frac{x N_d}{\langle x, x N_d \rangle}$$

Since $\langle x, x P_d \rangle = \langle x, x N_d \rangle$.

$$P_d = \frac{x P_d}{\langle x, x N_d \rangle}$$

Then for any $x \in H$ we have

$$N_d \subseteq M_d$$

Theorem 4.38. Given two closed subspaces $M$ and $N$, the projection $P_d$ and $N_d$ satisfy $P_d \subseteq M$ if and only if $P_d \subseteq N$. 

**Chapter 4. Hilbert Spaces**

4.7. Order Relation on Self-Adjoint Operators—Proofs of Theorems