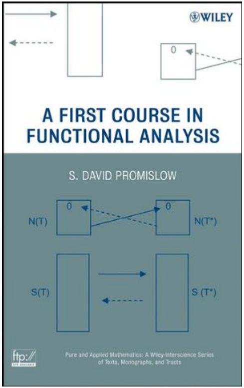
Introduction to Functional Analysis

Chapter 4. Hilbert Spaces

4.7. Order Relation on Self-Adjoint Operators—Proofs of Theorems



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$$\begin{aligned} \langle P_M x, x \rangle &= \langle P_M P_M x, x \rangle \text{ since } P_M^2 = P_M \\ &= \langle P_M x, P_M x \rangle \text{ since } P_M^* = P_M \\ &= \|P_M x\|^2 \| \\ &\leq \|P_N x\|^2 \text{ since } M \subseteq N \\ &= \langle P_N x, x \rangle \text{ (as above for } P_M). \end{aligned}$$

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So $\langle P_N x, x\rangle - \langle P_M x, x\rangle \geq 0$, or $\langle (P_N - P_M)x, x, \rangle \geq 0$ for all $x \in H$, and hence $P_M \leq P_N$.

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$$\begin{aligned} \langle P_n x, x \rangle &= \langle P_N x, P_N x \rangle \text{ (as above)} \\ &+ \|P_N x\|^2 < 1 \text{ since } x \notin N. \end{aligned}$$

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