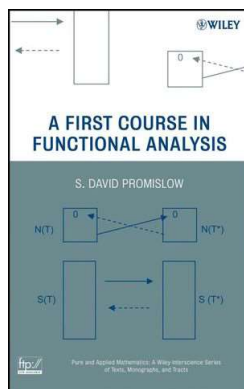


Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.2. Basic Version of Hahn-Banach Theorem—Proofs of Theorems



Theorem 5.1

Theorem 5.1. Hahn-Banach Extension Theorem.

Suppose that p is a Minkowski functional on a real linear space X and f_0 is a linear functional defined on a subspace Y of X such that $f_0(y) \leq p(y)$ for all $y \in Y$. Then, f_0 has an extension to a linear functional f defined on X such that $f(y) = f_0(y)$ for all $y \in Y$ and $f(x) \leq p(x)$ for all $n \in X$.

Proof (Stage 1). Let $x \in X$ where $x \notin Y$. We will extend f_0 from space Y to $\text{span}(Y \cup \{x\})$. Now $\text{span}(Y \cup \{x\})$ is the set of all vectors in X of the form $y = \alpha x$ for some $y \in Y$ and some $\alpha \in \mathbb{R}$ (recall that Y is a subspace). In fact, the choice of y and α are unique (since $x \notin Y$). We must choose $r \in \mathbb{R}$ such that $f(y + \alpha x) = f_0(y) + \alpha r$.

Define

$$A = \{f_0(y) - p(y - x) \mid y \in Y\}$$

and

$$B = \{p(y + x) - f_0(y) \mid y \in Y\}.$$

Theorem 5.1 (Stage 1 continued)

Proof (continued). Notice that for all $y, z \in Y$ we have that

$$\begin{aligned} f_0(y) + f_0(z) &= f_0(y + z) \text{ since } f_0 \text{ is linear on } Y \\ &\leq p(y + z) \text{ since } f_0 \leq p \text{ on } Y \\ &= p(y + x + z - x) \\ &\leq p(y + x) + p(z - x) \text{ since } p \text{ is a Minkowski functional.} \end{aligned}$$

So $f_0(y) - p(z - x) \leq p(y + x) - f_0(z)$ for all $y, z \in Y$. So any element of A is less than or equal to any element of B and $\sup(A) \leq \inf(B)$. Choose $r \in \mathbb{R}$ to satisfy $\sup(A) \leq r \leq \inf(B)$ (notice that r may not be unique). Therefore $f_0(y) - p(y - x) \leq r \leq p(y + x) - f_0(y)$ for all $y \in Y$, or $f_0(y) - r \leq p(y - x)$ and $f_0(y) + r \leq p(y + x)$. So we have

$$f_0(y) + \alpha r \leq p(y + \alpha x) \quad (*)$$

for all $y \in Y$ and for $\alpha = \pm 1$.

Theorem 5.1 (Stage 1 continued again)

Theorem 5.1. Hahn-Banach Extension Theorem.

Suppose that p is a Minkowski functional on a real linear space X and f_0 is a linear functional defined on a subspace Y of X such that $f_0(y) \leq p(y)$ for all $y \in Y$. Then, f_0 has an extension to a linear functional f defined on X such that $f(y) = f_0(y)$ for all $y \in Y$ and $f(x) \leq p(x)$ for all $n \in X$.

Proof (continued again). Then for any $\alpha > 0$, we have that

$$\begin{aligned} f_0(y) \pm \alpha r &= \alpha((1/\alpha)f_0(y) \pm r) \\ &= \alpha(f_0(y/\alpha) \pm r) \text{ since } f_0 \text{ is linear on } Y \\ &\leq \alpha p(y/\alpha \pm r) \text{ from } (*) \\ &= p(y \pm \alpha r). \end{aligned}$$

So we have that $f_0(y) + \alpha r \leq p(y + \alpha r)$ for all $\alpha \in \mathbb{R}$. Hence, with $f(y + \alpha x) = f_0(y) + \alpha r$, we have that $f \leq p$ on $\text{span}(Y \cup \{x\})$.

Theorem 5.1 (Stage 2)

Theorem 5.1. Hahn-Banach Extension Theorem.

Suppose that p is a Minkowski functional on a real linear space X and f_0 is a linear functional defined on a subspace Y of X such that $f_0(y) \leq p(y)$ for all $y \in Y$. Then, f_0 has an extension to a linear functional f defined on X such that $f(y) = f_0(y)$ for all $y \in Y$ and $f(x) \leq p(x)$ for all $x \in X$.

Proof (Stage 2). Define the set

$$(Z, g) = \{(Z, g) \mid Z \text{ is a subspace of } X \text{ containing } Y,$$

$g \text{ is a linear functional on } Z \text{ that extends to } f_0 \text{ and } f_0 \leq p \text{ on } Z\}.$

Then $(Y, f_0) \in (Z, g)$, so $(Z, g) \neq \emptyset$. Define a partial order on (Z, g) as $(Z_1, g_1) \leq (Z_2, g_2)$ if Z_1 is a subspace of Z_2 and g_2 extends to g_1 . Let \mathcal{V} be a linearly ordered subset of (Z, g) . Let $W = \cup\{V \mid (V, g) \in \mathcal{V}\}$ and define h on W as $h(x) = g(x)$ if $x \in V$ where $(V, g) \in \mathcal{V}$. Notice that h is well-defined since for $x \in V_1 \cap V_2$ where (say) $(V_1, g_1) \leq (V_2, g_2)$ then $h(x) = g_1(x) = g_2(x)$ since g_2 extends g_1 .

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Theorem 5.1 (Stage 2 continued)

Theorem 5.1. Hahn-Banach Extension Theorem.

Suppose that p is a Minkowski functional on a real linear space X and f_0 is a linear functional defined on a subspace Y of X such that $f_0(y) \leq p(y)$ for all $y \in Y$. Then, f_0 has an extension to a linear functional f defined on X such that $f(y) = f_0(y)$ for all $y \in Y$ and $f(x) \leq p(x)$ for all $x \in X$.

Proof (Stage 2 continued). Since all g 's are linear functionals, then h is a linear functional (that extends f_0). W is a subspace of X by Exercise 1.7. Then (W, h) is an upper bound for \mathcal{V} since $(V_1, g) \leq (W, h)$ for all $(V, g) \in \mathcal{V}$. So, by Zorn's Lemma, there is a maximal element of (Z, g) . If this maximal element is not defined on all of X , or if there is $x \in X$ with $x \notin Z$, then applying Stage 1 to $\text{span}(Z \cup \{x\})$ would yield a strictly "greater" element of (Z, g) , contradicting maximality. So the maximal element yields the desired extension f on X . \square

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