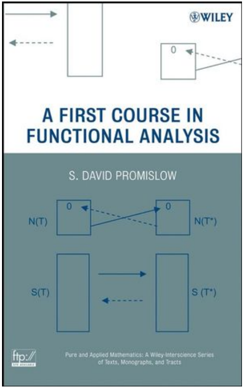
Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.2. Basic Version of Hahn-Banach Theorem—Proofs of Theorems



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Proof (Stage 1). Let $x \in X$ where $x \notin Y$. We will extend f_0 from space Y to span$(Y \cup \{x\})$.

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Define

$$A = \{f_0(y) - p(y - x) \mid y \in Y\}$$

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Proof (continued). Notice that for all $y, z \in Y$ we have that

$$\begin{aligned} f_0(y) + f_0(z) &= f_0(y+z) \text{ since } f_0 \text{ is linear on } Y \\ &\leq p(y+z) \text{ since } f_0 \leq p \text{ on } y \\ &= p(y+x+z-x) \\ &\leq p(y+x) + p(z-x) \text{ since } p \text{ is a Minkowski functional.} \end{aligned}$$

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So $f_0(y) - p(z-x) \leq p(y+x) - f_0(z)$ for all $y, z \in Y$. So any element of A is less than or equal to any element of B and $\sup(A) \leq \inf(B)$.

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$$f_0(y) + \alpha r \leq p(y + \alpha x) \qquad (*)$$

for all $y \in Y$ and for $\alpha = \pm 1$.

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$$\begin{aligned} f_0(y) \pm \alpha r &= \alpha((1/\alpha)f_0(y) \pm r) \\ &= \alpha(f_0(y/\alpha) \pm r) \text{ since } f_0 \text{ is linear on } Y \\ &\leq \alpha p(y/\alpha \pm r) \text{ from } (*) \\ &= p(y \pm \alpha r). \end{aligned}$$

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Proof (Stage 2). Define the set

$$(Z, g) = \{(Z, g) \mid Z \text{ is a subspace of } X \text{ containing } Y,$$

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