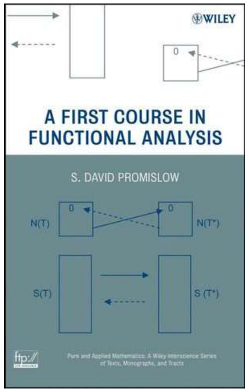
Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.3. Complex Version of the Hahn-Banach Theorem—Proofs of Theorems



Proposition 5.2

Proposition 5.2. A function $f : X \to \mathbb{C}$ is in $X^{\mathbb{C}}$ (i.e., f is a complex valued linear functional) if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both linear real valued functionals on X and, for all $x \in X$, $\mathrm{Im}(f(x)) = -\mathrm{Re}(f(ix))$.

Proof. We have $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$ and if f is linear, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are linear and for all $x \in X$, $f(ix) = if(x)$ or $\mathrm{Re}(f(ix)) + i\mathrm{Im}(f(ix)) = i\mathrm{Re}(f(x)) - \mathrm{Im}(f(x))$ and so $\mathrm{Im}(f(x)) = -\mathrm{Re}(f(ix))$.
Conversely, suppose $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both linear real valued functionals and for all $x \in X$, $\mathrm{Im}(f(x)) = -\mathrm{Re}(f(x))$. Then for real scalars a and b,

$$\begin{aligned} f(ax_1 + bx_2) &= \mathrm{Re}(f(zx_1 + bx_2)) + i\mathrm{Im}(f(zx_1 + bx_2)) \\ &= \mathrm{Re}(af(x_1) + bf(x_2)) + i\mathrm{Im}(af(x_1) + bf(x_2)) \\ &= a\mathrm{Re}(f(x_1)) + b\mathrm{Re}(f(x_2)) + ai\mathrm{Im}(f(x_1)) + bi\mathrm{Im}(f(x_2)) \\ &= \cdots = f(ax_1 + bx_2). \end{aligned}$$

Proposition 5.2 (continued)

Proposition 5.2. A function $f : X \to \mathbb{C}$ is in $X^{\mathbb{C}}$ (i.e., f is a complex valued linear functional) if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both linear real valued functionals on X and, for all $x \in X$, $\mathrm{Im}(f(x)) = -\mathrm{Re}(f(ix))$.

Proof (continued). For complex scalars, we need only consider $f(ix)$ and show this equals $if(x)$. We have

$$\begin{aligned} f(ix) &= \mathrm{Re}(f(ix)) + i\mathrm{Im}(f(ix)) \\ &= -\mathrm{Im}(f(x)) + i\mathrm{Im}(f(ix)) \text{ since } \mathrm{Im}(f(x)) = -\mathrm{Re}(f(ix)) \\ &= -\mathrm{Im}(f(x)) + i\mathrm{Re}(f(x)) \text{ since } \mathrm{Im}(f(ix)) = -\mathrm{Re}(f(i^2x)) \\ &\qquad\qquad\qquad\qquad = -\mathrm{Re}(f(-x)) = \mathrm{Re}(f(x)) \\ &= i(\mathrm{Re}(f(x)) + i\mathrm{Im}(f(x)) \\ &= if(x). \end{aligned}$$

Therefore f is linear over all complex scalars. □

Theorem 5.3. The Complex Hahn-Banach Extension Theorem

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Proof. By Proposition 5.2, $\mathrm{Re}(f_0(y))$ is a real valued linear functional on Y and $\mathrm{Re}(f_0(y)) \le |\mathrm{Re}(f_0(y))| \le |f_0(y)| \le \|y\|$ for all $y \in Y$, so by the Hahn-Banach Extension Theorem (Theorem 5.1), there is a real linear functional g on X which extends $\mathrm{Re}(f_0)$ and satisfies $g(x) \le \|x\|$ for all $x \in X$. Define $f : X \to \mathbb{C}$ as $f(x) = g(x) + ig(-ix)$. Since $\mathrm{Im}(f(x)) = g(-ix) = -g(ix) = -\mathrm{Re}(f(ix))$, then by Proposition 5.2, f is linear on X. Given $x \in X$, write $f(x) = re^{i\theta}$ where $r \ge 0$.

Theorem 5.3 (continued)

Theorem 5.3. Complex Hahn-Banach Extension Theorem.
Suppose $\|\cdot\|$ is a seminorm on a complex linear space X and that f_0 is a linear functional defined on a subspace Y of X such that $|f_0(y)| \leq \|y\|$ for all $y \in Y$. Then f_0 has an extension to a linear functional f on X such that $|f(x)| \leq \|x\|$ for all $x \in X$.

Proof (continued). Then $|f(x)| = r = e^{-i\theta} r e^{i\theta} = e^{-i\theta} f(x) = f(e^{-i\theta} x)$ since f is linear. Since $|f(x)|$ is real, then $f(e^{-i\theta}x)$ is real and so for all $x \in X$,

$$\begin{aligned} |f(x)| = f(e^{-i\theta}x) &= \mathrm{Re}(f(e^{-i\theta}x)) = g(e^{-i\theta}x) \\ &\leq \|e^{i\theta}x\| \text{ by the bound on } g \\ &= \|x\|. \end{aligned}$$

So $|f(x)| \leq \|x\|$ for all $x \in X$. □