Therefore f is linear over all complex scalars.

\[
\begin{align*}
(\lambda_1 f)(x) + (\lambda_2 f)(x) &= \lambda_1 f(x) + \lambda_2 f(x) \\
&= \lambda_1 (x) + \lambda_2 (x) \\
&= \lambda_1 (x) + \lambda_2 (x) \\
&= (\lambda_1 + \lambda_2) f(x) \\
\end{align*}
\]

That is, \( f \) is linear over all complex scalars. We have shown this equals \( f(x) \). For complex scalars, we need only consider \( f(x) \) and \( f(x) \).

**Theorem 5.3: The Complex Hahn-Banach Extension Theorem**

**Proposition 5.2 (continued):** A function \( f : X \rightarrow \mathbb{C} \) is in \( X \) if and only if \( \Re(f) \) and \( \Im(f) \) are both linear real valued functionals.

\[
\begin{align*}
(\Re(f))(x) &= (x)(f) \\
(\Im(f))(x) &= (x)(f) \\
\end{align*}
\]
So for all \( x \in \mathcal{X} \),

\[
\|x\| = \|x_{\theta \varepsilon}\| \leq \|x\|
\]

Since \( f \) is linear, \( f(x_{\theta \varepsilon}) = (f(x))_{\theta \varepsilon} \) is real, and so for all \( x \in \mathcal{X} \),

\[
(f(x))_{\theta \varepsilon} = f(x)_{\theta \varepsilon} = f(x) = |(x)|
\]

Proof (continued). Then that for all \( y \in \mathcal{X} \),

\[
|\langle x, y \rangle| = |(x)_{\theta \varepsilon} y_{\theta \varepsilon}| = |f(x)_{\theta \varepsilon} y_{\theta \varepsilon}| = |f(x) y_{\theta \varepsilon}| = |(x) y_{\theta \varepsilon}|
\]

Define a complex linear functional on a complex linear space \( \mathcal{X} \) and that \( g \) is a linear functional defined on a subspace \( \mathcal{Y} \) of such that for all \( y \in \mathcal{Y} \),

\[
|\langle x, y \rangle| \leq \|x\| \|y\|
\]

Suppose that for all \( x \in \mathcal{X} \),

\[
\|x\| \geq |(x)|
\]

Theorem 5.3. Complex Hahn-Banach Extension Theorem (continued)