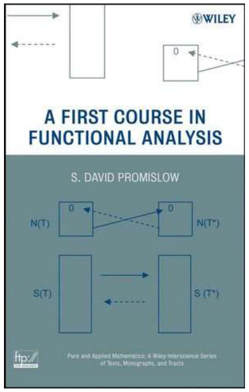
Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.4. Application to Normed Linear Spaces—Proofs of Theorems



Theorem 5.4

Theorem 5.4. Normed Linear Space Version of Hahn-Banach Extension Theorem.
Suppose that f_0 is a bounded linear functional defined on a subspace Y of a normed linear space X. Then f_0 has an extension to a bounded linear functional f on X such that $\|f\| = \|f_0\|$.

Proof. Define the seminorm (in fact, norm) $\|x\|_s = \|f_0\|\|x\|$. By the Complex Hahn-Banach Extension Theorem (Theorem 5.3), since $|f_0(y)| \le \|y\|_s = \|f_0\|\|y\|$ (by the definition of $\|f_0\|$), there is a linear functional f on X such that for all $x \in X$, $|f(x)| \le \|x\|_s = \|f_0\|\|x\|$. So $\|f\| \le \|f_0\|$. Since f is an extension of f_0, then $\|f\| \ge \|f_0\|$ (by the definition of the operator norm). So $\|f\| = \|f_0\|$. □

Corollary 5.5

Corollary 5.5. Given any closed subspace Y of a normed linear space X and $x \notin Y$, there is a bounded linear functional f on X (i.e., $f \in X^*$) such that $f(Y) = 0$ and $f(x) = 1$.

Proof. Define f_0 on $\text{span}(Y \cup \{x\})$ as $f_0(y + \alpha x) = \alpha$ for all $y \in Y$ and all scalars $\alpha \in \mathbb{F}$. The nullspace of f_0 is Y (the nullspace results when $\alpha = 0$). Y is, by hypothesis, closed in X, and so closed in any subspace containing Y (such as $\text{span}(Y \cup \{x\})$). Also, $f(x) = 1$ since $x = y + \alpha x$ for $y = 0$ and $\alpha = 1$. By the Normed Linear Space Version of Hahn-Banach Extension Theorem (Theorem 5.4), there is a bounded extension f of f_0 (on $\text{span}(Y \cup \{x\})$) to all of X with $\|f\| = \|f_0\|$. □

Corollary 5.6

Corollary 5.6. Given a normed linear space X and two points $x \ne y$, there is a bounded linear functional (i.e., $f \in X^*$) such that $f(x) \ne f(y)$.

Proof. Consider the subspace $\{0\}$ of X and the point $x - y \notin \{0\}$. By Corollary 5.5, there is a bounded linear f on X such that $f(x - y) = f(x) - f(y) = 1$ and so $f(x) \ne f(y)$. □

Corollary 5.7

Corollary 5.7. Consider linear space X with dual space X^*. The closed unit ball in X^* consists of all bounded linear functionals on X of functional norm less than or equal to 1. For any $x \in X$, we have

$$\|x\| = \sup\{|f(x)| \mid f \text{ is in the closed unit ball of } X^*\}.$$

Proof. Since $|f(x)| \leq \|f\|\|x\|$ by definition of $\|f\|$ and we have $\|f\| \leq 1$, then $|f(x)| \leq \|x\|$, and so $\|x\| \geq \sup\{|f(x)| \mid \|f\| \leq 1\}$. For $x \in X$, define f_0 on span$(\{x\})$ as $f_0(\alpha x) = \alpha\|x\|$ for scalar $\alpha \in \mathbb{F}$. Then $\|f_0\| = 1$ since $|f_0(y)| = \|y\|$ for all $y \in \text{span}(\{x\})$. By the Normed Linear Space Version of Hahn-Banach Extension Theorem (Theorem 5.4), f_0 extends to f on X where $\|f\| = \|f_0\| = 1$. So for our $x \in X$ we have $|f(x)| = \|f\|\|x\| = \|x\|$ and so $\sup\{|f(x)| \mid \|f\| \leq 1\| \geq \|x\|$. Equality follows. □