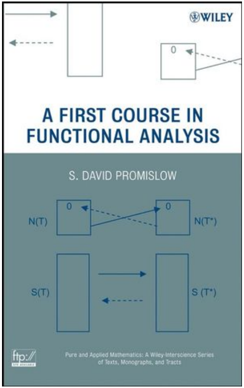
Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.4. Application to Normed Linear Spaces—Proofs of Theorems



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Suppose that f_0 is a bounded linear functional defined on a subspace Y of a normed linear space X. Then f_0 has an extension to a bounded linear functional f on X such that $\|f\| = \|f_0\|$.

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Proof. Define the seminorm (in fact, norm) $\|x\|_s = \|f_0\|\|x\|$. By the Complex Hahn-Banach Extension Theorem (Theorem 5.3), since $|f_0(y)| \leq \|y\|_s = \|f_0\|\|y\|$ (by the definition of $\|f_0\|$), there is a linear functional f on X such that for all $x \in X$, $|f(x)| \leq \|x\|_s = \|f_0\|\|x\|$. So $\|f\| \leq \|f_0\|$.

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Corollary 5.5

Corollary 5.5. Given any closed subspace Y of a normed linear space X and $x \notin Y$, there is a bounded linear functional f on X (i.e., $f \in X^*$) such that $f(Y) = 0$ and $f(x) = 1$.

Proof. Define f_0 on $\text{span}(Y \cup \{x\})$ as $f_0(y + \alpha x) = \alpha$ for all $y \in Y$ and all scalars $\alpha \in \mathbb{F}$.

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Corollary 5.6

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Proof. Consider the subspace $\{0\}$ of X and the point $x - y \notin \{0\}$. By Corollary 5.5, there is a bounded linear f on X such that $f(x - y) = f(x) - f(y) = 1$ and so $f(x) \neq f(y)$. $\square$

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Corollary 5.7. Consider linear space X with dual space X^*. The closed unit ball in X^* consists of all bounded linear functionals on X of functional norm less than or equal to 1. For any $x \in X$, we have

$$\|x\| = \sup\{|f(x)| \mid f \text{ is in the closed unit ball of } X^*\}.$$

Proof. Since $|f(x)| \leq \|f\|\|x\|$ by definition of $\|f\|$ and we have $\|f\| \leq 1$, then $|f(x)| \leq \|x\|$, and so $\|x\| \geq \sup\{|f(x)| \mid \|f\| \leq 1\}$.

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