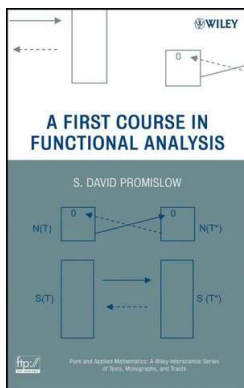


# Introduction to Functional Analysis

## Chapter 5. Hahn-Banach Theorem

### 5.5. Geometric Version of Hahn-Banach Theorem—Proofs of Theorems



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Lemma 1

## Lemma 1 (continued)

**Lemma 1.** Let  $H$  be a hyperplane in  $X$ . If  $H = f^{-1}(\{\alpha\})$  and  $H = g^{-1}(\{\beta\})$  for some linear functionals  $f$  and  $g$ , then  $f = \gamma g$  for some  $\gamma \in \mathbb{R}$ .

**Proof (continued).** If  $x' = 0$  and  $H$  is a subspace, then  $H = f^{-1}(0) = g^{-1}(0)$ . Let  $x \in X$  and  $z \notin H$ . Since  $H$  is maximal,  $X = \text{span}(Y \cup \{z\})$  and for any  $y + \delta z \in X$  we have, as above,  $f(y + \delta z) = \delta f(z)$  and  $g(y + \delta z) = \delta g(z)$ . So for any  $x \in X$  we have  $g(x) = (g(z)/f(z))f(x)$  (where  $g(z)$  and  $f(z)$  are constant).  $\square$

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Lemma 1

## Lemma 1

**Lemma 1.** Let  $H$  be a hyperplane in  $X$ . If  $H = f^{-1}(\{\alpha\})$  and  $H = g^{-1}(\{\beta\})$  for some linear functionals  $f$  and  $g$ , then  $f = \gamma g$  for some  $\gamma \in \mathbb{R}$ .

**Proof.** Suppose  $x' + Y = H = f^{-1}(\{\alpha\}) = g^{-1}(\{\beta\})$ . Then for any  $v, w \in Y$ ,  $\beta = g(x' + v) = g(x' + w)$ , and  $g(v) = g(w) = \beta - g(x')$  and so  $g$  is constant on  $Y$ . Since  $0 \in Y$  ( $Y$  is a subspace),  $g(0) = 0$  and so  $g(y) = 0$  for all  $y \in Y$ . So if  $x' \neq 0$  then  $g(x') = g(x' + 0) = \beta$ . Similarly,  $f(x') = \alpha$ . Since for  $x' \notin H$ ,  $\text{span}(T \cup \{x'\}) = X$  and for any  $y + \delta x' \in X$  we have

$$f(y + \delta x') = f(y) + \delta f(x') = \delta f(x') = \delta \alpha,$$

$$g(y + \delta x') = g(y) + \delta g(x') = \delta g(x') = \delta \beta.$$

So for any  $x \in X$  we have  $f(x) = (\alpha/\beta)g(x)$ .

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Proposition 5.9

## Proposition 5.9

**Proposition 5.9.** For a convex set in  $\mathbb{R}^n$ , all internal points are interior.

**Proof.** By Theorem 2.31(b), all norms on  $\mathbb{R}^n$  are equivalent, so we choose the sup norm as the norm on  $\mathbb{R}^n$ , without loss of generality. Let  $a$  be an internal point of a convex set  $A \subset \mathbb{R}^n$ . Recall that  $\delta_i$  represents the  $i$ th standard basis vector for  $\mathbb{R}^n$ . Since  $a$  is an internal point of  $A$  then for all vectors  $x \in \mathbb{R}^n$  there exists  $r_x > 0$  such that  $a + tr_x \in A$  for  $0 \leq t \leq r_x$  (by definition). So we consider the parameters  $r_{\delta_i}$  and  $r_{-\delta_i}$  for each standard basis vector and its negative and define  $r = \min\{r_{\delta_i}, r_{-\delta_i} \mid i = 1, 2, \dots, n\}$ . Then  $r > 0$  (this is where finite dimensional is needed). Let  $v$  be any vector in  $\mathbb{R}^n$  with norm  $\|v\| < r/n$ ; say  $v = \sum_{i=1}^n \alpha_i \delta_i$ . Then

$$a + v = \frac{1}{n} \sum_{i=1}^n (a + n\alpha_i \delta_i).$$

Now each  $|\alpha_i| \leq r/n$ , or  $n|\alpha_i| \leq r \leq r_{\delta_i}$  and  $n|\alpha_i| \leq r \leq r_{-\delta_i}$  (we need to consider the “direction”  $-\delta_i$  since  $\alpha_i$  may be negative).

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## Proposition 5.9 (continued)

**Proposition 5.9.** For a convex set in  $\mathbb{R}^n$ , all internal points are interior.

**Proof (continued).** Therefore  $a + n\alpha_i\delta_i \in A$  for each  $i = 1, 2, \dots, n$ . The convexity of  $A$  then implies, by Exercise 5.18(a), that

$$a + v = \sum_{i=1}^n \frac{1}{n}(a + n\alpha_i\delta_i) \in A.$$

Since this holds for all  $v \in \mathbb{R}^n$  with  $\|v\| < r/n$ , then  $B(a; r/n) \subset A$ . Therefore,  $a$  is an interior point of  $A$ .  $\square$

## Proposition 5.10

**Proposition 5.10.** If  $A$  is convex, then  $A^{\circ\circ} = A^\circ$ .

**Proof.** Let  $a \in A^\circ$  and for each unit vector  $y$  define  $r_y > 0$  such that  $a + ty \in A$  for all  $0 \leq t \leq r_y$ . We now show that each  $a + ty$  for  $0 \leq t \leq r_y/2$  is itself internal to  $A^\circ$ . Let  $z$  be a unit vector and define  $r_z$  as above. Then for  $0 \leq s \leq r_z/2$  and for  $0 \leq t \leq r_y/2$  we have  $(a + ty) + sz = \frac{1}{2}((a + 2ty) + (a + 2sz))$ . Now  $a + 2ty \in A$  for  $0 \leq t \leq r_y/2$  and  $a + 2sz \in A$  for  $0 \leq s \leq r_z/2$ , so  $\frac{1}{2}((a + 2ty) + (a + 2sz)) \in A$  since  $A$  is convex. So  $(a + ty) \in A^\circ$  for all  $0 \leq t \leq r_y/2$ . That is,  $a \in A^{\circ\circ}$ . So  $A^\circ \subseteq A^{\circ\circ}$ . Of course,  $A^{\circ\circ} \subseteq A^\circ$ , and hence  $A^\circ = A^{\circ\circ}$ .  $\square$

## Proposition 5.11

**Proposition 5.11.** Given a Minkowski functional  $p$ , let  $K_p = \{x \mid p(x) < 1\}$ . Then  $K_p$  is convex and  $0$  is an internal point of  $K_p$ .

**Proof.** Let  $x, y \in K_p$  and  $\alpha \in [0, 1]$ . Then

$$\begin{aligned} p(\alpha x + (1 - \alpha)y) &\leq p(\alpha x) + p((1 - \alpha)y) \\ &= \alpha p(x) + (1 - \alpha)p(y) < \alpha(1) + (1 - \alpha)(1) = 1, \end{aligned}$$

so  $K_p$  is convex.

Next, fix  $x \in X$  and choose  $r_x > 0$  such that  $r_x p(x) < 1$ . Then, for  $0 \leq t \leq r_x$ ,

- (1) if  $p(x) \geq 0$  then  $p(0 + tx) = p(tx) = tp(x) \leq r_x p(x) < 1$ ,
- (2) if  $p(x) \leq 0$  then  $p(0 + tx) = p(tx) = tp(x) \leq 0 < 1$ .

So  $0 + tx \in K_p$  for  $0 \leq t \leq r_x$ , and  $0$  is internal to  $K_p$ .  $\square$

## Proposition 5.12

**Proposition 5.12.** In a real normed linear space with a given convex set  $K$  which has  $0$  as an internal point. Define

$$p(x) = \inf\{t > 0 \mid x/t \in K\}.$$

Then:

- (a)  $p$  is a Minkowski functional,
- (b)  $p(x) < 1$  if and only if  $x$  is an internal point of  $K$ ,
- (c)  $p(x) = 1$  if and only if  $x$  is a bounding point of  $K$ , and
- (d)  $p(x) > 1$  if and only if  $x$  is an external point of  $K$ .

## Proposition 5.12(a)

**Proposition 5.12.** In a real normed linear space with a given convex set  $K$  which has 0 as an internal point. Define  $p(x) = \inf\{t > 0 \mid x/t \in K\}$ . Then:

(a)  $p$  is a Minkowski functional.

**Proof.** For  $\alpha \geq 0$  we have

$$p(\alpha x) = \inf\{t > 0 \mid \alpha x/t \in K\} = \alpha \inf\{t > 0 \mid x/t \in K\} = \alpha p(x).$$

Given any  $x, y \in K$ , suppose  $x/t$  and  $y/s$  are in  $K$  for some  $t > 0$  and  $s > 0$ . Then

$$\frac{x+y}{t+s} = \left(\frac{t}{t+s}\right) \frac{x}{t} + \left(\frac{s}{t+s}\right) \frac{y}{s} \in K$$

since  $K$  is convex (and the coefficients in parentheses sum to 1). So  $p(x+y) = \inf\{r > 0 \mid (x+y)/r \in K\} \leq t+s$ . Taking infima over  $t$  and  $s$  such that  $x/t \in K$  and  $y/s \in K$  we get  $p(x+y) \leq p(x) + p(y)$ .  $\square$

## Proposition 5.12(b)

**Proposition 5.12.** In a real normed linear space with a given convex set  $K$  which has 0 as an internal point. Define  $p(x) = \inf\{t > 0 \mid x/t \in K\}$ . Then:

(b)  $p(x) < 1$  if and only if  $x$  is an internal point of  $K$ .

**Proof of (b).** If  $x$  is internal to  $K$ , choose  $t > 0$  such that  $x + tx = (1+t)x \in K$ . This shows that  $p(x) \leq 1/(1+t) < 1$ . Conversely, suppose  $p(x) < 1$ . Given any  $y \in X$ , choose  $r_y$  so that  $p(x) + r_y p(y) < 1$ . Then for  $0 \leq t \leq r_y$  we have

$$\begin{aligned} p(x+ty) &\leq p(x) + p(ty) \text{ since } p \text{ is a Minkowski functional by (a)} \\ &= p(x) + tp(y) \leq p(x) + r_y p(y) < 1. \end{aligned}$$

So, by Lemma 2,  $x + ty \in K$ . Therefore,  $x$  is internal to  $K$ .  $\square$

## Proposition 5.12(d)

**Proposition 5.12.** In a real normed linear space with a given convex set  $K$  which has 0 as an internal point. Define  $p(x) = \inf\{t > 0 \mid x/t \in K\}$ . Then:

(d)  $p(x) > 1$  if and only if  $x$  is an external point of  $K$ .

**Proof of (d).** If  $x$  is external to  $K$ , choose  $0 < t < 1$  such that  $x - tx = (1-t)x \notin K$ . By Lemma 2,  $p((1-t)x) = (1-t)p(x) \geq 1$ . So  $p(x) \geq 1/(1-t) > 1$ .

Conversely, suppose that  $p(x) > 1$ . For any  $y \in X$ , choose  $r_y > 0$  so that  $p(x) - r_y p(-y) > 1$ . Then for  $0 \leq t \leq r_y$  we have

$$p(x) = p(x + ty - ty) \leq p(x + ty) + tp(-y) \quad (*)$$

so that

$$p(x + ty) \geq p(x) - tp(-y) \text{ by } (*)$$

## Proposition 5.12(d) continued, and (c)

**Proposition 5.12.** In a real normed linear space with a given convex set  $K$  which has 0 as an internal point. Define  $p(x) = \inf\{t > 0 \mid x/t \in K\}$ . Then:

(c)  $p(x) = 1$  if and only if  $x$  is a bounding point of  $K$ , and

(d)  $p(x) > 1$  if and only if  $x$  is an external point of  $K$ .

**Proof of (d) continued.** ... so that

$$\begin{aligned} p(x+ty) &\geq p(x) - tp(-y) \text{ by } (*) \\ &\geq p(x) - r_y p(-y) \text{ since } -t \geq -r_y \\ &> 1. \end{aligned}$$

By Lemma 2,  $x + ty \notin K$ . So  $x$  is an internal point of  $K^c = X \setminus K$ . That is,  $x$  is an external point of  $K$ .  $\square$

**Proof of (c).** This follows from (b) and (d) and the definition of boundary point.  $\square$

## Proposition 5.13

**Proposition 5.13.** Let  $K$  be a convex set which has some internal point and let  $f$  be a real valued linear functional on  $X$ . Then  $f(K^\circ)$  is the interior of the interval  $f(K)$ .

**Proof.** Since  $K$  is convex, for any  $x, y \in K$  we have  $\alpha x + (1 - \alpha)y \in K$  for  $\alpha \in [0, 1]$ . So

$$f(\alpha x + (1 - \alpha)y) \in f(K), \text{ or } \alpha f(x) + (1 - \alpha)f(y) \in f(K).$$

So  $f(K)$  is convex and since  $f(K) \subseteq \mathbb{R}$ , then  $f(K)$  is an interval,  $I$ . Let  $a = \inf(I)$  and  $b = \sup(I)$ . We will show that  $f(K^\circ) = (a, b)$ . By translation, we can translate an internal point to 0, so without loss of generality 0 is an internal point.

Suppose  $x \in K^\circ$ . Choose  $t > 0$  such that  $x + tx = (1 + t)x \in K$ . Then

$$f(x + tx) = f((1 + t)x) = (1 + t)f(x) \in f(K) = I.$$

So  $(1 + t)f(x) \leq b$  and  $f(x) < b$ . Similarly choose  $t > 0$  such that  $x + t(-x) = (1 - t)x \in K$ .

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## Proposition 5.13 (Part 2)

**Proof (Part 2).** Then

$$f(x + t(-x)) = f((1 - t)x) = (1 - t)f(x) \in f(K) = I.$$

So  $(1 + t)f(x) \leq b$  and  $f(x) < b$ . Similarly choose  $t > 0$  such that  $x + t(-x) = (1 - t)x \in f(K) = I$ . So  $(1 - t)f(x) \geq a$  and  $f(x) < a$ . So  $f(x) \in I$ .

Conversely, suppose  $c \in (a, b)$ . If  $c \geq 0$ , pick  $c_1$  such that  $c < c_1 < b$ , and choose  $x \in K$  such that  $f(x) = c_1$ . Let  $z = (c/c_1)x$ . Then  $f(z) = f((c/c_1)x) = \frac{c}{c_1}f(x) = \frac{c}{c_1}c_1 = c$ . Also,  $p_K(x) \leq 1$  since  $x \in K$  (by Proposition 5.12) and so

$$\begin{aligned} p_K(z) &= p_K((c/c_1)x) = (c/c_1)p_K(x) \\ &< 1 \text{ since } c/c_1 < 1 \text{ and } p_K(x) < 1. \end{aligned}$$

So  $z$  is an internal point of  $K$  by Proposition 5.12.

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## Proposition 5.13 (Part 3)

**Proposition 5.13.** Let  $K$  be a convex set which has some internal point and let  $f$  be a real valued linear functional on  $X$ . Then  $f(K^\circ)$  is the interior of the interval  $f(K)$ .

**Proof (Part 3).** If  $c < 0$ , pick  $c_2$  such that  $a < c_2 < c < 0$  and choose  $x \in K$  such that  $f(x) = c_2$ . Let  $z = (c/c_2)x$ . Then  $f(z) = f((c/c_2)x) = (c/c_2)f(x) = (c/c_2)c_2 = c$ . Also,  $p_K(x) \leq 1$  since  $x \in K$  (by Proposition 5.12), and so

$$\begin{aligned} p_K(z) &= p_K((c/c_2)x) = (c/c_2)p_K(x) \\ &< 1 \text{ since } c_2 < c < 0 \text{ implies } 1 > c/c_2 > 0 \text{ and } p_K(x) \leq 1. \end{aligned}$$

Again,  $z$  is an internal point of  $K$  by Proposition 5.12. So  $f(K^\circ) = (a, b) = (f(K))^\circ$ .  $\square$

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## Proposition 5.14

**Theorem 5.14.** *Geometric Hahn-Banach Extension Theorem.*

We consider a real normed linear space. Let  $K$  be a convex set with an internal point, and let  $P$  be a linear manifold such that  $P \cap K^\circ = \emptyset$ . Then there is a hyperplane  $H$  containing  $P$  such that  $H \cap K^\circ = \emptyset$ .

**Proof.** Without loss of generality, 0 is an internal point of  $K$  (since everything can be translated such that the internal point is translated to 0). Let  $P = x_0 + Y$  be the linear manifold (where  $Y$  is a subspace). We have  $x_0 \neq 0$  since we hypothesized  $P \cap K^\circ = \emptyset$  and  $0 \in K^\circ$ . Define  $f_0$  a real valued linear functional on  $\text{span}(Y \cup \{x_0\})$  as  $f_0(y + \alpha x_0) = \alpha$  for all  $y \in Y$  and  $\alpha \in \mathbb{R}$ . Then for any  $y \in Y$  we have  $y + x_0 \notin K^\circ$  since  $P \cap K^\circ = \emptyset$ , so by Proposition 5.12 we have  $f_0(y + x_0) = 1 \leq p_K(y + x_0)$ . Also,  $f_0(y - x_0) = -1 \leq p_K(y - x_0)$  (since  $p_K \geq 0$ ). So we have  $f_0(y + \alpha x_0) \leq p_K(y + \alpha x_0)$  for any  $y \in Y$  and for  $\alpha = \pm 1$ . As in the proof of Theorem 5.1, this is sufficient to show that for all  $z \in \text{span}(Y \cup \{x_0\})$  we have  $f_0(z) \leq p_K(z)$ .

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## Theorem 5.14 (continued)

**Theorem 5.14.** *Geometric Hahn-Banach Extension Theorem.*

We consider a real normed linear space. Let  $K$  be a convex set with an internal point, and let  $P$  be a linear manifold such that  $P \cap K^\circ = \emptyset$ . Then there is a hyperplane  $H$  containing  $P$  such that  $H \cap K^\circ = \emptyset$ .

**Proof (continued).** We can extend  $f_0$  to a linear functional  $f$  on all of  $X$  such that  $f$  is dominated by  $p_K$ , by the Hahn-Banach Extension Theorem (Theorem 5.1). Let  $H = f^{-1}(\{1\})$ . Then  $H$  contains all of  $P$  (since  $P = 1x_0 + Y$  and  $f$  maps all of  $P$  to 1), and  $H$  is a hyperplane by the note above (or the text on page 109). For all  $x \in K^\circ$ , we have  $f(x) \leq p_K(x) < 1$ , so  $H \cap K^\circ$  is empty.  $\square$

## Theorem 5.15

**Theorem 5.15. The Hahn-Banach Separation Theorem.**

We consider a real normed linear space. Let  $K$  and  $L$  be convex sets such that  $K$  has some internal point and  $L \cap K^\circ = \emptyset$ . Then there is a hyperplane separating  $K$  and  $L$ .

**Proof.** By Proposition 5.12,  $K^\circ = p_K^{-1}[0, 1)$ . So for  $x, y \in K^\circ$  we have for  $\alpha \in [0, 1]$ ,

$$p_K(\alpha x + (1 - \alpha)y) = \alpha p_K(x) + (1 - \alpha)p_K(y) < \alpha(1) + (1 - \alpha)(1) = 1.$$

So  $\alpha x + (1 - \alpha)y \in K^\circ$  and  $K^\circ$  is convex. Consider the set  $J = K^\circ - L$ . Let  $j_1, j_2 \in J$ . Then  $j_1 = k_1 - \ell_1$  and  $j_2 = k_2 - \ell_2$  for some  $k_1, k_2 \in K^\circ$  and some  $\ell_1, \ell_2 \in L$ . So for any  $\alpha \in [0, 1]$ ,

$$\begin{aligned} \alpha j_1 + (1 - \alpha)j_2 &= \alpha(k_1 - \ell_1) + (1 - \alpha)(k_2 - \ell_2) \\ &= (\alpha k_1 + (1 - \alpha)k_2) - (\alpha \ell_1 + (1 - \alpha)\ell_2) \in K^\circ - L, \end{aligned}$$

and so  $J = K^\circ - L$  is convex.

## Theorem 5.15 (continued)

**Proof (continued).** First, we show  $J^\circ = J$ . Of course,  $J^\circ \subseteq J$ . Let  $k - \ell \in J$  where  $k \in K^\circ$  and  $\ell \in L$ . Since  $K^{\circ\circ} = K^\circ$  (Proposition 5.10),  $k$  is internal to  $K^\circ$ , so for all  $x \in X$  there is  $r_x > 0$  such that for  $0 \leq \alpha \leq r_x$  we have  $k + \alpha x \in K^\circ$ . Therefore  $(k + \alpha x) - \ell = (k - \ell) + \alpha x \in K^\circ - L = J$ , or  $k - \ell \in J^\circ$ . So  $J \subseteq J^\circ$  and  $J^\circ = J$ .

Since we hypothesize that  $L \cap K^\circ = \emptyset$ , then  $0 \notin K^\circ - L = J$ . Consider  $\{0\}$  as a linear manifold; by the Geometric Hahn-Banach Extension Theorem (Theorem 5.14), there is a hyperplane  $H$  containing 0 and not intersecting  $J = J^\circ$ . Since  $H$  is a hyperplane containing 0, then there is a linear functional  $f$  on the whole space such that the nullspace  $N(f) = H$  (see page 109). So  $N(f) \cap J = \emptyset$ . As in the proof of Proposition 5.13,  $f(K^\circ)$  and  $f(L)$  are intervals of real numbers. These intervals are disjoint because, otherwise  $f(k) = f(\ell)$  for some  $k \in K^\circ$  and  $\ell \in L$  and then  $f(k - \ell) = f(k) - f(\ell) = 0$ , implying  $k - \ell \in N(f)$ , but  $k - \ell \in J = K - L$  and  $N(f) \cap J = \emptyset$  as shown above.

## Theorem 5.15 (continued again)

**Theorem 5.15. The Hahn-Banach Separation Theorem.**

We consider a real normed linear space. Let  $K$  and  $L$  be convex sets such that  $K$  has some internal point and  $L \cap K^\circ = \emptyset$ . Then there is a hyperplane separating  $K$  and  $L$ .

**Proof (continued again).** For any two disjoint intervals, there is some real number between the two. Possibly by replacing  $f$  with  $-f$  (which preserves all the needed properties of  $f$ ;  $-f$  is linear and  $N(f) = N(-f)$ ) there is  $c > 0$  such that  $f(K^\circ) < c \leq f(L)$ . In Proposition 5.13 it is shown that  $f(K)$  and  $f(K^\circ)$  are both intervals of real numbers and differ at most by the inclusion/exclusion of endpoints. So we have  $f(K) \leq c \leq f(L)$ , since  $f(K)$  and  $f(K^\circ)$  can only differ by endpoints. So the hyperplane  $f^{-1}(\{c\})$  separates  $f(K)$  and  $f(L)$ , since they lie in different half spaces. (Since we do not have a strict inequality, we may not have a strict separation and open half spaces).  $\square$

## Theorem 5.16

**Theorem 5.16.** We consider a real normed linear space. Let  $K$  be a convex set with an internal point such that  $K$  contains all its bounding points. Then  $K$  is the intersection of all the half spaces containing  $K$  that are determined by the supporting hyperplanes.

**Proof.** By translation, we can assume  $0$  is internal to  $K$ . Let  $K_1$  be the intersection of all supporting hyperplanes containing  $K$ . Since all these hyperplanes contain  $K$ , then  $K \subseteq K_1$ . Suppose  $y \notin K$ . Let  $x = y/p_K(y)$ . Then  $p_K(x) = 1$  and so by Proposition 5.12,  $x$  is a boundary point of  $K$ . So (by the Hahn-Banach Separation Theorem, Theorem 5.15) there is a supporting hyperplane  $H$  to  $K$  through  $x$ .

Choose a linear functional  $f$  such that  $H = f^{-1}(\alpha)$  for some given  $\alpha \in \mathbb{R}$  (as can be done by the comments on page 109). By replacing  $f$  with  $-f$  if necessary, we can have  $f(K) \subseteq (-\infty, \alpha]$  (recall that  $f(K)$  is an interval of real numbers as shown in Proposition 5.13).

## Theorem 5.16 (continued)

**Theorem 5.16.** We consider a real normed linear space. Let  $K$  be a convex set with an internal point such that  $K$  contains all its bounding points. Then  $K$  is the intersection of all the half spaces containing  $K$  that are determined by the supporting hyperplanes.

**Proof (continued).** Since  $0 \in K$  and  $0 \in f^{-1}(\{0\}) = N(f)$ , then  $\alpha \geq 0$ . Since  $x \in H$ , then  $f(x) = \alpha$ . Since  $y \notin K$  and  $K$  contains all its boundary points, then  $y$  is not a boundary point of  $K$ . So  $p_K(y) > 1$  by Proposition 5.12, and hence

$$f(y) = f(p_K(y)x) = p_K(y)f(x) = \alpha p_K(y) > \alpha.$$

So  $y$  is not in the half space containing  $K$  determined by  $H$  (since  $H = f^{-1}(\{\alpha\})$  and  $f(K) \subseteq (-\infty, \alpha]$ ). So  $y \notin K_1$ . We have shown the contrapositive of  $y \in K_1 \Rightarrow y \in K$ . That is,  $K_1 \subseteq K$ . Therefore  $K = K_1$ . □

## Theorem 5.17

**Theorem 5.17.** If  $K$  and  $L$  are disjoint convex sets of a real normed linear space  $X$ , where  $K$  is compact and  $L$  is closed, then there is a hyperplane strictly separating  $K$  and  $L$ .

**Proof.** As argued in the proof of the Hahn-Banach Separation Theorem (Theorem 5.15), the difference of two convex sets is convex, so the set  $K - L$  is convex. By Lemma 3,  $K - L$  is closed. Since  $K$  and  $L$  are disjoint,  $0 \notin K - L$  and since  $K - L$  is closed, for some  $r > 0$  we have  $B(r) \cap (K - L) = \emptyset$ . So  $K - L$  is convex and  $B(r)$  is convex with an internal point and  $B(r)^\circ = B(r)$ . So by applying the Hahn-Banach Separation Theorem (Theorem 5.15), there is a hyperplane separating  $K - L$  and  $B(r)^\circ = B(r)$ . So there is a linear functional  $f$  such that the interval in  $\mathbb{R}$  of  $f(K - L)$  (we get intervals in  $\mathbb{R}$  as seen in the proof of Proposition 5.13) is disjoint from the interval in  $\mathbb{R}$  of  $f(B(r))$ . (The disjointness follows from the strict inequality in the first displaymath equation on page 115). Since  $0 = f(0) \in f(B(r))$ , we can assume (possibly by replacing  $f$  with  $-f$ ) that  $f(K - L) \subseteq (0, \infty)$ .

## Theorem 5.17 (continued)

**Theorem 5.17.** If  $K$  and  $L$  are disjoint convex sets of a real normed linear space  $X$ , where  $K$  is compact and  $L$  is closed, then there is a hyperplane strictly separating  $K$  and  $L$ .

**Proof (continued).** Let  $\beta = \sup f(L)$  and  $\gamma = \inf f(K)$ . It must be that  $\beta < \gamma$ , or else we could find  $k \in K$  and  $\ell \in L$  such that  $\|k - \ell\|$  is arbitrarily small, but then we could have  $k - \ell \in B(r)$ , or  $B(r) \cap (K - L) \neq \emptyset$ , a contradiction. Let  $\delta$  satisfy  $\beta < \delta < \gamma$ . Then  $f^{-1}(\{\delta\})$  is a hyperplane separating  $K$  and  $L$  since  $f(K) \subseteq (-\infty, \beta]$  and  $f(L) \subseteq [\gamma, \infty)$ . □

## Theorem 5.18

**Theorem 5.18.** Let  $K$  and  $L$  be two disjoint convex sets in  $\mathbb{R}^n$ . Then there is a hyperplane  $H$  separating  $K$  and  $L$ .

**Proof.** Without loss of generality (by translation) we may assume  $0 \in K$ . First, suppose  $\dim(K) = \dim(\text{span}(K)) = n$ . Then by Exercise 5.18,  $K$  contains an internal point. By the Hahn-Banach Separation Theorem (Theorem 5.15) there is a hyperplane separating  $K$  and  $L$ . Second, suppose  $\dim(K) = \dim(\text{span}(K)) = m < n$  and suppose the result holds for  $\dim(K) \in \{m+1, m+2, \dots, n\}$  (the text calls the technique used here “backwards induction”). Choose  $x \notin \text{span}(K)$ . Let  $K_+ = \{k + \alpha x \mid k \in K, \alpha \in [0, 1]\}$  and  $K_- = \{k - \alpha x \mid k \in K, \alpha \in [0, 1]\}$ . Both  $K_+$  and  $K_-$  are convex and  $\dim(K_+) = \dim(K_-) = m+1$ . Notice that set  $L$  cannot intersect both  $K_+$  and  $K_-$ , or else we could find  $\ell_1, \ell_2 \in L$ ,  $k_1, k_2 \in K$ , and  $\alpha_1, \alpha_2 \in [0, 1]$  such that  $\ell_1 = k_1 + \alpha_1 x$  and  $\ell_2 = k_2 - \alpha_2 x$ .

## Theorem 5.18 (continued)

**Theorem 5.18.** Let  $K$  and  $L$  be two disjoint convex sets in  $\mathbb{R}^n$ . Then there is a hyperplane  $H$  separating  $K$  and  $L$ .

**Proof (continued).** Then for  $\beta = \alpha_2 / (\alpha_1 + \alpha_2) \in [0, 1]$  and  $1 - \beta = \alpha_1 / (\alpha_1 + \alpha_2)$  we have  $\beta \ell_1 = \alpha_2(k_1 + \alpha_1 x) / (\alpha_1 + \alpha_2)$  and  $(1 - \beta)\ell_2 = \alpha_1(k_2 - \alpha_2 x) / (\alpha_1 + \alpha_2)$  and so

$$\begin{aligned} \beta \ell_1 + (1 - \beta)\ell_2 &= \frac{\alpha_1(k_1 + \alpha_1 x)}{\alpha_1 + \alpha_2} + \frac{\alpha_2(k_2 - \alpha_2 x)}{\alpha_1 + \alpha_2} \\ &= \alpha_1 k_1 / (\alpha_1 + \alpha_2) + \alpha_2 k_2 / (\alpha_1 + \alpha_2) = \beta k_1 + (1 - \beta)k_2. \end{aligned}$$

But since  $\beta \in [0, 1]$  and  $K$  and  $L$  are convex, then  $\beta \ell_1 + (1 - \beta)\ell_2 = \beta k_1 + (1 - \beta)k_2$  is in both  $L$  and  $K$ , contradicting the hypothesis of disjointness. Therefore, either  $L \cup K_- = \emptyset$  or  $L \cap K_+ = \emptyset$ . Without loss of generality, suppose  $L \cap K_- = \emptyset$ . Then by the induction hypothesis, there is a hyperplane  $H$  that separates  $K_-$  and  $L$ . Since  $K \subseteq K_-$ , then  $H$  separates  $K$  and  $L$ . The result now follows by “backwards induction.” □