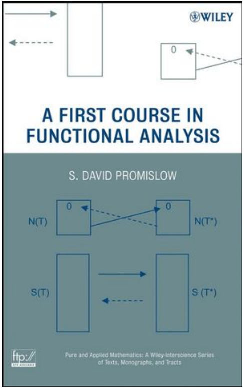
Introduction to Functional Analysis

Chapter 5. Hahn-Banach Theorem

5.5. Geometric Version of Hahn-Banach Theorem—Proofs of Theorems



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Lemma 1. Let H be a hyperplane in X. If $H = f^{-1}(\{\alpha\})$ and $H = g^{-1}(\{\beta\})$ for some linear functionals f and g, then $f = \gamma g$ for some $\gamma \in \mathbb{R}$.

Proof. Suppose $x' + Y = H = f^{-1}(\{\alpha\}) = g^{-1}(\{\beta\})$.

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Proof. Suppose $x' + Y = H = f^{-1}(\{\alpha\}) = g^{-1}(\{\beta\})$. Then for any $v, w \in Y$, $\beta = g(x' + v) = g(x' + w)$, and $g(v) = g(w) = \beta - g(x')$ and so g is constant on Y. Since $0 \in Y$ (Y is a subspace), $g(0) = 0$ and so $g(y) = 0$ for all $y \in Y$. So if $x' \neq 0$ then $g(x') = g(x' + 0) = \beta$. Similarly, $f(x') = \alpha$. Since for $x' \notin H$, $\mathrm{span}(T \cup \{x'\}) = X$ and for any $y + \delta x' \in X$ we have

$$f(y + \delta x') = f(y) + \delta f(x') = \delta f(x') = \delta\alpha,$$

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So for any $x \in X$ we have $f(x) = (\alpha/\beta) g(x)$.

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Proof (continued). If $x' = 0$ and H is a subspace, then $H = f^{-1}(0) = g^{-1}(0)$. Let $x \in X$ and $z \notin H$. Since H is maximal, $X = \text{span}(Y \cup \{z\})$ and for any $y + \delta z \in X$ we have, as above, $f(y + \delta z) = \delta f(z)$ and $g(y + \delta z) = \delta g(z)$.

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Proposition 5.9

Proposition 5.9. For a convex set in $\mathbb{R}^n$, all internal points are interior.

Proof. By Theorem 2.31(b), all norms on $\mathbb{R}^n$ are equivalent, so we choose the sup norm as the norm on $\mathbb{R}^n$, without loss of generality. Let a be an internal point of a convex set $A \subset \mathbb{R}^n$. Recall that δ_i represents the ith standard basis vector for $\mathbb{R}^n$. Since a is an internal point of A then for all vectors $x \in \mathbb{R}^n$ there exists $r_x > 0$ such that $a + tr_x \in A$ for $0 \leq t \leq r_x$ (by definition).

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$$a + v = \frac{1}{n} \sum_{i=1}^{n} (a + n\alpha_i \delta_i).$$

Now each $|\alpha_i| \leq r/n$, or $n|\alpha_i| \leq r \leq r_{\delta_i}$ and $n|\alpha_i| \leq r \leq r_{-\delta_i}$ (we need to consider the "direction" $-\delta_i$ since α_i may be negative).

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Proof (continued). Therefore $a + n\alpha_i\delta_i \in A$ for each $i = 1, 2, \ldots, n$. The convexity of A then implies, by Exercise 5.18(a), that

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Since this holds for all $v \in \mathbb{R}^n$ with $\|v\| < r/n$, then $B(a; r/n) \subset A$. Therefore, a is an interior point of A. $\square$

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Proposition 5.10

Proposition 5.10. If A is convex, then $A^{oo} = A^{o}$.

Proof. Let $a \in A^{o}$ and for each unit vector y define $r_y > 0$ such that $a + ty \in A$ for all $0 \leq t \leq r_y$. We now show that each $a + ty$ for $0 \leq t \leq r_y/2$ is itself internal to A^{o}.

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Proof. Let $x, y \in K_p$ and $\alpha \in [0, 1]$.

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$$p(\alpha x + (1-\alpha)y) \leq p(\alpha x) + p((1-\alpha)y)$$

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Next, fix $x \in X$ and choose $r_x > 0$ such that $r_x p(x) < 1$. Then, for $0 \leq t \leq r_x$,

(1) if $p(x) \geq 0$ then $p(0 + tx) = p(tx) = tp(x) \leq r_x p(x) < 1$,

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So $0 + tx \in K_p$ for $0 \leq t \leq r_x$, and 0 is internal to K_p. □

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Proposition 5.12

Proposition 5.12. In a real normed linear space with a given convex set K which has 0 as an internal point. Define

$$p(x) = \inf\{t > 0 \mid x/t \in K\}.$$

Then:

(a) p is a Minkowski functional,

(b) $p(x) < 1$ if and only if x is an internal point of K,

(c) $p(x) = 1$ if and only if x is a bounding point of K, and

(d) $p(x) > 1$ if and only if x is an external point of K.

Proposition 5.12(a)

Proposition 5.12. In a real normed linear space with a given convex set K which has 0 as an internal point. Define $p(x) = \inf\{t > 0 \mid x/t \in K\}$. Then:

(a) p is a Minkowski functional.

Proof. For $\alpha \geq 0$ we have

$$p(\alpha x) = \inf\{t > 0 \mid \alpha x/t \in K\} = \alpha \inf\{t > 0 \mid x/t \in K\} = \alpha p(x).$$

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Proof. For $\alpha \geq 0$ we have

$$p(\alpha x) = \inf\{t > 0 \mid \alpha x/t \in K\} = \alpha \inf\{t > 0 \mid x/t \in K\} = \alpha p(x).$$

Given any $x, y \in K$, suppose x/t and y/s are in K for some $t > 0$ and $s > 0$. Then

$$\frac{x+y}{t+s} = \left(\frac{t}{t+2}\right)\frac{x}{t} + \left(\frac{s}{t+s}\right)\frac{y}{s} \in K$$

since K is convex (and the coefficients in parentheses sum to 1).

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since K is convex (and the coefficients in parentheses sum to 1). So $p(x+y) = \inf\{r > 0 \mid (x+y)/r \in K\} \leq t+s$.

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since K is convex (and the coefficients in parentheses sum to 1). So $p(x+y) = \inf\{r > 0 \mid (x+y)/r \in K\} \leq t + s$. Taking infima over t and s such that $x/t \in K$ and $y/s \in K$ we get $p(x+y) \leq p(x) + p(y)$. □

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$$p(\alpha x) = \inf\{t > 0 \mid \alpha x/t \in K\} = \alpha \inf\{t > 0 \mid x/t \in K\} = \alpha p(x).$$

Given any $x, y \in K$, suppose x/t and y/s are in K for some $t > 0$ and $s > 0$. Then

$$\frac{x+y}{t+s} = \left(\frac{t}{t+2}\right)\frac{x}{t} + \left(\frac{s}{t+s}\right)\frac{y}{s} \in K$$

since K is convex (and the coefficients in parentheses sum to 1). So $p(x+y) = \inf\{r > 0 \mid (x+y)/r \in K\} \leq t+s$. Taking infima over t and s such that $x/t \in K$ and $y/s \in K$ we get $p(x+y) \leq p(x) + p(y)$. □

Proposition 5.12(b)

Proposition 5.12. In a real normed linear space with a given convex set K which has 0 as an internal point. Define $p(x) = \inf\{t > 0 \mid x/t \in K\}$. Then:

(b) $p(x) < 1$ if and only if x is an internal point of K.

Proof of (b). If x is internal to K, choose $t > 0$ such that $x + tx = (1 + t)x \in K$. This shows that $p(x) \leq 1/(1 + t) < 1$.

Proposition 5.12(b)

Proposition 5.12. In a real normed linear space with a given convex set K which has 0 as an internal point. Define $p(x) = \inf\{t > 0 \mid x/t \in K\}$. Then:

(b) $p(x) < 1$ if and only if x is an internal point of K.

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$$\begin{aligned} p(x + ty) &\leq p(x) + p(ty) \text{ since } p \text{ is a Minkowski functional by (a)} \\ &= p(x) + tp(y) \leq p(x) + r_y p(y) < 1. \end{aligned}$$

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Proof of (d). If x is external to K, choose $0 < t < 1$ such that $x - tx = (1-t)x \notin K$. By Lemma 2, $p((1-t)x) = (1-t)p(x) \geq 1$. So $p(x) \geq 1/(1-t) > 1$.

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$$p(x) = p(x + ty - ty) \leq p(x + ty) + tp(-y) \qquad (*)$$

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Proposition 5.12(d) continued, and (c)

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(c) $p(x) = 1$ if and only if x is a bounding point of K, and

(d) $p(x) > 1$ if and only if x is an external point of K.

Proof of (d) continued. ...so that

$$\begin{aligned} p(x+ty) &\geq p(x) - tp(-y) \text{ by } (*) \\ &\geq p(x) - r_y p(-y) \text{ since } -t \geq -r_y \\ &> 1. \end{aligned}$$

By Lemma 2, $x + ty \notin K$. So x is an internal point of $K^c = X \setminus K$. That is, x is an external point of K. □

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Proof. Since K is convex, for any $x, y \in K$ we have $\alpha x + (1 - \alpha)y \in K$ for $\alpha \in [0, 1]$.

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$$f(\alpha x + (1-\alpha)y) \in f(K), \text{ or } \alpha f(x) + (1-\alpha)f(y) \in f(K).$$

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So $(1+t)f(x) \leq b$ and $f(x) < b$. Similarly choose $t > 0$ such that $x + t(-x) = (1-t)x \in K$.

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Proof (Part 2). Then

$$f(x + t(-x)) = f((1-t)x) = (1-t)f(x) \in f(x) = I.$$

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$$\begin{aligned} p_K(z) &= p_K((c/c_1)x) = (c/c_1)p_K(x) \\ &< 1 \text{ since } c/c_1 < 1 \text{ and } p_K(x) < 1. \end{aligned}$$

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Proposition 5.13. Let K be a convex set which has some internal point and let f be a real valued linear functional on X. Then $f(K^o)$ is the interior of the interval $f(K)$.

Proof (Part 3). If $c < 0$, pick c_2 such that $a < c_2 < c < 0$ and choose $x \in K$ such that $f(x) = c_2$. Let $z = (c/c_2)x$. Then $f(z) = f((c/c_2)x) = (c/c_2)f(x) = (c/c_2)c_2 = c$. Also, $p_K(x) \leq 1$ since $x \in K$ (by Proposition 5.12), and so

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Proposition 5.14

Theorem 5.14. *Geometric Hahn-Banach Extension Theorem.*
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

Proof. Without loss of generality, 0 is an internal point of K (since everything can be translated such that the internal point is translated to 0). Let $P = x_0 + Y$ be the linear manifold (where Y is a subspace). We have $x_0 \neq 0$ since we hypothesized $P \cap K^o = \varnothing$ and $0 \in K^o$.

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Define f_0 a real valued linear functional on $\text{span}(Y \cup \{x_0\})$ as $f_0(y + \alpha x_0) = \alpha$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Then for any $y \in Y$ we have $y + x_0 \notin K^o$ since $P \cap K^o = \varnothing$, so by Proposition 5.12 we have $f_0(y + x_0) = 1 \leq p_K(y + x_0)$.

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Theorem 5.14. *Geometric Hahn-Banach Extension Theorem.*
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

Proof. Without loss of generality, 0 is an internal point of K (since everything can be translated such that the internal point is translated to 0). Let $P = x_0 + Y$ be the linear manifold (where Y is a subspace). We have $x_0 \neq 0$ since we hypothesized $P \cap K^o = \varnothing$ and $0 \in K^o$.
Define f_0 a real valued linear functional on $\text{span}(Y \cup \{x_0\})$ as $f_0(y + \alpha x_0) = \alpha$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Then for any $y \in Y$ we have $y + x_0 \notin K^o$ since $P \cap K^o = \varnothing$, so by Proposition 5.12 we have $f_0(y + x_0) = 1 \leq p_K(y + x_0)$. Also, $f_0(y - x_0) = -1 \leq p_K(y - x_0)$ (since $p_K \geq 0$). So we have $f_0(y + \alpha x_0) \leq p_K(y + \alpha x_0)$ for any $y \in Y$ and for $\alpha = \pm 1$. As in the proof of Theorem 5.1, this is sufficient to show that for all $z \in \text{span}(Y \cup \{x_0\})$ we have $f_0(z) \leq p_K(z)$.

Theorem 5.14 (continued)

Theorem 5.14. *Geometric Hahn-Banach Extension Theorem.*
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

Proof (continued). We can extend f_0 to a linear functional f on all of X such that f is dominated by p_K, by the Hahn-Banach Extension Theorem (Theorem 5.1).

Theorem 5.14 (continued)

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Proof (continued). We can extend f_0 to a linear functional f on all of X such that f is dominated by p_K, by the Hahn-Banach Extension Theorem (Theorem 5.1). Let $H = f^{-1}(\{1\})$. Then H contains all of P (since $P = 1x_0 + Y$ and f maps all of P to 1), and H is a hyperplane by the note above (or the text on page 109).

Theorem 5.14 (continued)

Theorem 5.14. *Geometric Hahn-Banach Extension Theorem.*
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

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Theorem 5.14 (continued)

Theorem 5.14. *Geometric Hahn-Banach Extension Theorem.*
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

Proof (continued). We can extend f_0 to a linear functional f on all of X such that f is dominated by p_K, by the Hahn-Banach Extension Theorem (Theorem 5.1). Let $H = f^{-1}(\{1\})$. Then H contains all of P (since $P = 1x_0 + Y$ and f maps all of P to 1), and H is a hyperplane by the note above (or the text on page 109). For all $x \in K^o$, we have $f(x) \leq p_K(x) < 1$, so $H \cap K^o$ is empty. □

Theorem 5.15

Theorem 5.15. The Hahn-Banach Separation Theorem.
We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Proof. By Proposition 5.12, $K^o = p_K^{-1}[0,1)$.

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We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Proof. By Proposition 5.12, $K^o = p_K^{-1}[0,1)$. So for $x, y \in K^o$ we have for $\alpha \in [0,1]$,

$$p_K(\alpha x + (1-\alpha)y) = \alpha p_K(x) + (1-\alpha)p_K(y) < \alpha(1) + (1-\alpha)(1) = 1.$$

So $\alpha x + (1-\alpha)y \in K^o$ and K^o is convex.

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So $\alpha x + (1-\alpha)y \in K^o$ and K^o is convex. Consider the set $J = K^o - L$. Let $j_1, j_2 \in J$. Then $j_1 = k_1 - \ell_1$ and $j_2 = k_2 - \ell_2$ for some $k_1, k_2 \in K^o$ and some $\ell_1, \ell_2 \in L$.

Theorem 5.15

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We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Proof. By Proposition 5.12, $K^o = p_K^{-1}[0,1)$. So for $x, y \in K^o$ we have for $\alpha \in [0,1]$,

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So $\alpha x + (1-\alpha)y \in K^o$ and K^o is convex. Consider the set $J = K^o - L$. Let $j_1, j_2 \in J$. Then $j_1 = k_1 - \ell_1$ and $j_2 = k_2 - \ell_2$ for some $k_1, k_2 \in K^o$ and some $\ell_1, \ell_2 \in L$. So for any $\alpha \in [0,1]$,

$$\alpha j_1 + (1-\alpha)j_2 = \alpha(k_1 - \ell_1) + (1-\alpha)(k_2 - \ell_2)$$

$$= (\alpha k_1 + (1-\alpha)k_2) - (\alpha\ell_1 + (1-\alpha)\ell_2) \in K^o - L,$$

and so $J = K^o - L$ is convex.

Theorem 5.15

Theorem 5.15. The Hahn-Banach Separation Theorem.
We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Proof. By Proposition 5.12, $K^o = p_K^{-1}[0,1)$. So for $x, y \in K^o$ we have for $\alpha \in [0,1]$,

$$p_K(\alpha x + (1-\alpha)y) = \alpha p_K(x) + (1-\alpha)p_K(y) < \alpha(1) + (1-\alpha)(1) = 1.$$

So $\alpha x + (1-\alpha)y \in K^o$ and K^o is convex. Consider the set $J = K^o - L$. Let $j_1, j_2 \in J$. Then $j_1 = k_1 - \ell_1$ and $j_2 = k_2 - \ell_2$ for some $k_1, k_2 \in K^o$ and some $\ell_1, \ell_2 \in L$. So for any $\alpha \in [0,1]$,

$$\alpha j_1 + (1-\alpha) j_2 = \alpha(k_1 - \ell_1) + (1-\alpha)(k_2 - \ell_2)$$

$$= (\alpha k_1 + (1-\alpha)k_2) - (\alpha \ell_1 + (1-\alpha)\ell_2) \in K^o - L,$$

and so $J = K^o - L$ is convex.

Theorem 5.15 (continued)

Proof (continued). First, we show $J^o = J$. Of course, $J^o \subseteq J$. Let $k - \ell \in J$ where $k \in K^o$ and $\ell \in L$.

Theorem 5.15 (continued)

Proof (continued). First, we show $J^o = J$. Of course, $J^o \subseteq J$. Let $k - \ell \in J$ where $k \in K^o$ and $\ell \in L$. Since $K^{oo} = K^o$ (Proposition 5.10), k is internal to K^o, so for all $x \in X$ there is $r_x > 0$ such that for $0 \leq \alpha \leq r_x$ we have $k + \alpha x \in K^o$. Therefore

$(k + \alpha x) - \ell = (k - \ell) + \alpha x \in K^o - L = J$, or $k - \ell \in J^o$. So $J \subseteq J^o$ and $J^o = J$.

Theorem 5.15 (continued)

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Since we hypothesize that $L \cap K^o = \varnothing$, then $0 \notin K^o - L = J$.

Theorem 5.15 (continued)

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Since we hypothesize that $L \cap K^o = \varnothing$, then $0 \notin K^o - L = J$. Consider $\{0\}$ as a linear manifold; by the Geometric Hahn-Banach Extension Theorem (Theorem 5.14), there is a hyperplane H containing 0 and not intersecting $J = J^o$. Since H is a hyperplane containing 0, then there is a linear functional f on the whole space such that the nullspace $N(f) = H$ (see page 109). So $N(f) \cap J = \varnothing$.

Theorem 5.15 (continued)

Proof (continued). First, we show $J^o = J$. Of course, $J^o \subseteq J$. Let $k - \ell \in J$ where $k \in K^o$ and $\ell \in L$. Since $K^{oo} = K^o$ (Proposition 5.10), k is internal to K^o, so for all $x \in X$ there is $r_x > 0$ such that for $0 \le \alpha \le r_x$ we have $k + \alpha x \in K^o$. Therefore
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Theorem 5.15 (continued)

Proof (continued). First, we show $J^o = J$. Of course, $J^o \subseteq J$. Let $k - \ell \in J$ where $k \in K^o$ and $\ell \in L$. Since $K^{oo} = K^o$ (Proposition 5.10), k is internal to K^o, so for all $x \in X$ there is $r_x > 0$ such that for $0 \leq \alpha \leq r_x$ we have $k + \alpha x \in K^o$. Therefore
$(k + \alpha x) - \ell = (k - \ell) + \alpha x \in K^o - L = J$, or $k - \ell \in J^o$. So $J \subseteq J^o$ and $J^o = J$.

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Theorem 5.15 (continued again)

Theorem 5.15. The Hahn-Banach Separation Theorem.
We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Proof (continued again). For any two disjoint intervals, there is some real number between the two. Possibly by replacing f with $-f$ (which preserves all the needed properties of f; $-f$ is linear and $N(f) = N(-f)$) there is $c > 0$ such that $f(K^o) < c \leq f(L)$.

Theorem 5.15 (continued again)

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Theorem 5.15 (continued again)

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Theorem 5.16

Theorem 5.16. We consider a real normed linear space. Let K be a convex set with as internal point such that K contains all its bounding points. Then K is the intersection of all the half spaces containing K that are determined by the supporting hyperplanes.

Proof. By translation, we can assume 0 is internal to K. Let K_1 be the intersection of all supporting hyperplanes containing K. Since all these hyperplanes contain K, then $K \subseteq K_1$.

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Choose a linear functional f such that $H = f^{-1}(\alpha)$ for some given $\alpha \in \mathbb{R}$ (as can be done by the comments on page 109). By replacing f with $-f$ if necessary, we can have $f(K) \subseteq (-\infty, \alpha]$ (recall that $f(K)$ is an interval of real numbers as shown in Proposition 5.13).

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Proof (continued). Since $0 \in K$ and $0 \in f^{-1}(\{0\}) = N(f)$, then $\alpha \geq 0$. Since $x \in H$, then $f(x) = \alpha$. Since $y \notin K$ and K contains all its boundary points, then y is not a boundary point of K. So $p_K(y) > 1$ by Proposition 5.12, and hence

$$f(y) = f(p_K(y)x) = p_K(y)f(x) = \alpha p_+K(y) > \alpha.$$

Theorem 5.16 (continued)

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$$f(y) = f(p_K(y)x) = p_K(y)f(x) = \alpha p_+ K(y) > \alpha.$$

So y is not in the half space containing K determined by H (since $H = f^{-1}(\{\alpha\})$ and $f(K) \subseteq (-\infty, \alpha])$. So $y \notin K_1$.

Theorem 5.16 (continued)

Theorem 5.16. We consider a real normed linear space. Let K be a convex set with as internal point such that K contains all its bounding points. Then K is the intersection of all the half spaces containing K that are determined by the supporting hyperplanes.

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So y is not in the half space containing K determined by H (since $H = f^{-1}(\{\alpha\})$ and $f(K) \subseteq (-\infty, \alpha])$. So $y \notin K_1$. We have shown the contrapositive of $y \in K_1 \Rightarrow y \in K$. That is, $K_1 \subseteq K$. Therefore $K = K_1$. $\square$

Theorem 5.16 (continued)

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Proof (continued). Since $0 \in K$ and $0 \in f^{-1}(\{0\}) = N(f)$, then $\alpha \geq 0$. Since $x \in H$, then $f(x) = \alpha$. Since $y \notin K$ and K contains all its boundary points, then y is not a boundary point of K. So $p_K(y) > 1$ by Proposition 5.12, and hence

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Theorem 5.17

Theorem 5.17. If K and L are disjoint convex sets of a real normed linear space X, where K is compact and L is closed, then there is a hyperplane strictly separating K and L.

Proof. As argued in the proof of the Hahn-Banach Separation Theorem (Theorem 5.15), the difference of two convex sets is convex, so the set $K - L$ is convex. By Lemma 3, $K - L$ is closed.

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Proof. As argued in the proof of the Hahn-Banach Separation Theorem (Theorem 5.15), the difference of two convex sets is convex, so the set $K - L$ is convex. By Lemma 3, $K - L$ is closed. Since K and L are disjoint, $0 \notin K - L$ and since $K - L$ is closed, for some $r > 0$ we have $B(r) \cap (K - L) = \varnothing$. So $K - L$ is convex and $b(R)$ is convex with an internal point and $B(r)^o = B(r)$.

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Proof. As argued in the proof of the Hahn-Banach Separation Theorem (Theorem 5.15), the difference of two convex sets is convex, so the set $K - L$ is convex. By Lemma 3, $K - L$ is closed. Since K and L are disjoint, $0 \notin K - L$ and since $K - L$ is closed, for some $r > 0$ we have $B(r) \cap (K - L) = \varnothing$. So $K - L$ is convex and $b(R)$ is convex with an internal point and $B(r)^o = B(r)$. So by applying the Hahn-Banach Separation Theorem (Theorem 5.15), there is a hyperplane separating $K - L$ and $B(r)^o = B(r)$. So there is a linear functional f such that the interval in $\mathbb{R}$ of $f(K - L)$ (we get intervals in $\mathbb{R}$ as seen in the proof of Proposition 5.13) is disjoint from the interval in $\mathbb{R}$ of $f(B(r))$.

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Theorem 5.17. If K and L are disjoint convex sets of a real normed linear space X, where K is compact and L is closed, then there is a hyperplane strictly separating K and L.

Proof. As argued in the proof of the Hahn-Banach Separation Theorem (Theorem 5.15), the difference of two convex sets is convex, so the set $K - L$ is convex. By Lemma 3, $K - L$ is closed. Since K and L are disjoint, $0 \notin K - L$ and since $K - L$ is closed, for some $r > 0$ we have $B(r) \cap (K - L) = \varnothing$. So $K - L$ is convex and $b(R)$ is convex with an internal point and $B(r)^o = B(r)$. So by applying the Hahn-Banach Separation Theorem (Theorem 5.15), there is a hyperplane separating $K - L$ and $B(r)^o = B(r)$. So there is a linear functional f such that the interval in $\mathbb{R}$ of $f(K - L)$ (we get intervals in $\mathbb{R}$ as seen in the proof of Proposition 5.13) is disjoint from the interval in $\mathbb{R}$ of $f(B(r))$. (The disjointness follows from the strict inequality in the first displaymath equation on page 115). Since $0 = f(0) \in f(B(r))$, we can assume (possibly by replacing f with $-f$) that $f(K - L) \subseteq (0, \infty)$.

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Theorem 5.17 (continued)

Theorem 5.17. If K and L are disjoint convex sets of a real normed linear space X, where K is compact and L is closed, then there is a hyperplane strictly separating K and L.

Proof (continued). Let $\beta = \sup f(L)$ and $\gamma = \inf f(K)$. It must be that $\beta < \gamma$, or else we could find $k \in K$ and $\ell \in L$ such that $\|k - \ell\|$ is arbitrarily small, but then we could have $k - \ell \in B(r)$, or $B(r) \cap (K - L) \neq \varnothing$, a contradiction.

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Theorem 5.18

Theorem 5.18. Let K and L be two disjoint convex sets in $\mathbb{R}^n$. Then there is a hyperplane H separating K and L.

Proof. Without loss of generality (by translation) we may assume $0 \in K$.

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Proof. Without loss of generality (by translation) we may assume $0 \in K$. First, suppose $\dim(K) = \dim(\text{span}(K)) = n$. Then by Exercise 5.18, K contains an internal point. By the Hahn-Banach Separation Theorem (Theorem 5.15) there is a hyperplane separating K and L.

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Second, suppose $\dim(K) = \dim(\text{span}(K)) = m < n$ and suppose the result holds for $\dim(K) \in \{m+1, m+2, \ldots, n\}$ (the text calls the technique used here "backwards induction").

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Theorem 5.18 (continued)

Theorem 5.18. Let K and L be two disjoint convex sets in $\mathbb{R}^n$. Then there is a hyperplane H separating K and L.

Proof (continued). Then for $\beta = \alpha_2/(\alpha_1 + \alpha_2) \in [0, 1]$ and $1 - \beta = \alpha_1/(\alpha_1 + \alpha_2)$ we have $\beta\ell_1 = \alpha_2(k_1 + \alpha_1 x)/(\alpha_1 + \alpha_2)$ and $(1 - \beta)\ell_2 = \alpha_1(k_2 - \alpha_2 x)/(\alpha_1 + \alpha_2)$ and so

$$\beta\ell_1 + (1-\beta)\ell_2 = \frac{\alpha_1(k_1 + \alpha_1 x)}{\alpha_1 + \alpha_2} + \frac{\alpha_2(k_2 - \alpha_2 x)}{\alpha_1 + \alpha_2}$$

$$= \alpha_1 k_1/(\alpha_1 + \alpha_2) + \alpha_2 k_2/(\alpha_1 + \alpha_2) = \beta k_1 + (1-\beta)k_2.$$

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But since $\beta \in [0, 1]$ and K and L are convex, then $\beta\ell + (1-\beta)\ell_2 = \beta k_1 + (1-\beta)k_2$ is in both L and K, contradicting the hypothesis of disjointness. Therefore, either $L \cup K_- = \varnothing$ or $L \cap K_+ = \varnothing$. Without loss of generality, suppose $L \cap K_- = \varnothing$.

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Theorem 5.18 (continued)

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Proof (continued). Then for $\beta = \alpha_2/(\alpha_1+\alpha_2) \in [0,1]$ and $1-\beta = \alpha_1/(\alpha_1+\alpha_2)$ we have $\beta\ell_1 = \alpha_2(k_1+\alpha_1 x)/(\alpha_1+\alpha_2)$ and $(1-\beta)\ell_2 = \alpha_1(k_2-\alpha_2 x)/(\alpha_1+\alpha_2)$ and so

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