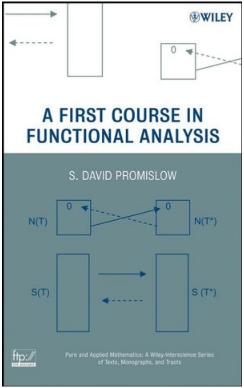
# Introduction to Functional Analysis

**Chapter 6. Duality**

6.1. Examples of Dual Spaces—Proofs of Theorems



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$$\begin{aligned} \|f_n - g\| &\geq |(f_n - g)(x_n)| \text{ by definition of operator norm} \\ &= |f_n(x_n) - g(x_n)| = |f_n(x_n)| \text{ since } g(x_n) = 0 \\ &\geq \|f_n\|/2 \text{ by the choice of } x_n. \end{aligned}$$

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# Theorem 6.2 (continued)

**Theorem 6.2.** If $X^*$ is separable, then $X$ is separable.

**Proof (continued).** Also,

$$\begin{aligned} 1 = \|g\| &= \|g - f_n + f_n\| \\ &\leq \|g - f_n\| + \|f_n\| \text{ by the Triangle Inequality} \\ &\leq \|f_n\|/2 + \|f_n\| \text{ by above} \\ &= \frac{3}{2}\|f_n\|. \end{aligned}$$

Therefore, $\|g - f_n\| \geq \frac{2}{3}\|f_n\|$. But then $\{f_n\}$ is not dense in $X^*$ (since no $f_n$ is "close to" $g$).

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