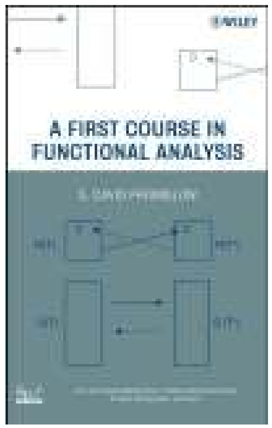
Introduction to Functional Analysis

Chapter 6. Duality

6.2. Adjoints—Proofs of Theorems



Theorem 6.6, Properties of the Adjoint in the Normed Linear Space Setting

Theorem 6.6. Properties of the Adjoint in the Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(a) $(S + T)^* = S^* + T^*$,
(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α),
(c) $(AT)^* = T^*A^*$, and
(d) $\|T^*\| = \|T\|$.

Proof of (a). Let $S, T \in \mathcal{B}(X, y)$ and $f \in Y^*$. Then for all $x \in X$

$$\begin{aligned} ((S+T)^*f)x &= f((S+Tx)) \text{ by the definition of adjoint} \\ &= f(Sx+Tx) \text{by definition of operator addition} \\ &= f(Sx) + f(Tx) \text{ since } f \text{ is linear} \\ &= (S^*f)(x) + (T^*f)(x) \text{ by definition of adjoint.} \end{aligned}$$

So $(S+T)^* = S^* + T^*$. □

Theorem 6.6(b)

Theorem 6.6. Properties of the Adjoint in the Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α).

Proof of (b). Let $T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$, and $f \in Y^*$. Then for all $x \in X$ we have

$$\begin{aligned} ((\alpha T)^*f)(x) &= f((\alpha T)x) \text{ by the definition of adjoint} \\ &= f(\alpha(Tx)) \text{ by the definition of scalar multiplication} \\ &\qquad \text{on an operator} \\ &= \alpha f(Tx) \text{ since } f \text{ is linear} \\ &= \alpha(T^*f)(x) \text{ by the definition of adjoint.} \end{aligned}$$

So $(\alpha T)^* = \alpha T^*$. □

Theorem 6.6(c)

Theorem 6.6. Properties of Adjoint in Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(c) $(AT)^* = T^*A^*$.

Proof of (c). Let $A \in \mathcal{B}(Y, Z)$, $T \in \mathcal{B}(X, Y)$, and $f \in Z^*$. We need to show that $(AT)^*f = T^*(A^*f)$. For all $x \in X$ we have

$$\begin{aligned} ((AT)^*f)(x) &= f((AT)(x)) \text{ by the definition of adjoint } (AT)^* \\ &\qquad (\text{notice that } (AT)(x) \in Z \text{ and } f \in Z^*) \\ &= f(A(T(x))) \\ &= (A^*f)(T(x)) \text{ by the definition of adjoint } A^* \\ &\qquad (\text{notice } T(x) \in Y \text{ and } Af \in Y^*) \\ &= (T^*(A^*f))(x) \text{ by definition of adjoint } T^* \text{ in terms of } A^*f \\ &\qquad (\text{notice } T^*(A^*f) \in X^* \text{ and so } T^*(A^*f) : X \to \mathbb{F}) \\ &= ((T^*A^*)f)(x) \text{ since operator composition is associative.} \end{aligned}$$

So $(AT)^* = T^*A^*$. □

Theorem 6.6(d)

Theorem 6.6. Properties of Adjoint in Normed Linear Space Setting. For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(d) $\|T^*\| = \|T\|$.

Proof of (d). We can scale to assume without loss of generality that $\|T\| = 1$. Then for any unit vectors $f \in Y^*$ and $x \in X$, we have

$$\begin{aligned} \|(T^*f)(x)\| &= |f(Tx)\| \text{ by the definition of adjoint } T^* \\ &\le \|f\|\|Tx\| \text{ by definition of norm of functional } f \\ &\le \|f\|\|T\|\|x\| \text{ by definition of } \|T\|, \text{ since } \|T\| = \|x\| = 1, \end{aligned}$$

so taking a supremum over all such $x \in X$, we have $\|T^*f\| \le 1$, and taking a supremum over all such f, we have $\|T^*\| \le 1$. Conversely, given any $\varepsilon > 0$, choose a unit vector $x \in X$ such that $\|Tx\| \ge 1 - \varepsilon$ (this can be done since $\|T\| = 1$). As seen in the proof of Corollary 5.7 (with Y^* replacing X and Y replacing X^*) there is a unit vector $f \in Y^*$ such that $|f(Tx)| = \|Tx\|$.

Theorem 6.6(d)

Theorem 6.6. Properties of Adjoint in Normed Linear Space Setting. For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(d) $\|T^*\| = \|T\|$.

Proof of (d) continued. So

$$\begin{aligned} |f(Tx)| &= |(T^*f)(x)| \text{ by the definition of } T^* \\ &= \|Tx\| \ge 1 - \varepsilon. \end{aligned}$$

Taking a supremum over all unit vectors $x \in X$ gives $\|fT\| \ge 1 - \varepsilon$. Taking a supremum over all unit vectors $f \in Y^*$ implies that $\|ft\| = \|T^*\| \ge 1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|T^*\| \ge 1$. Therefore $\|T^*\| = 1$, or $\|T^*\| = \|T\|$. $\square$