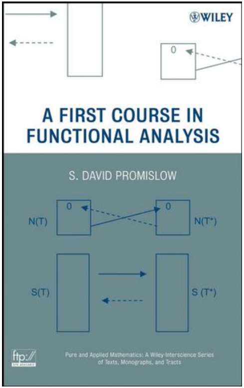
Introduction to Functional Analysis

Chapter 6. Duality

6.2. Adjoints—Proofs of Theorems



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For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(a) $(S + T)^* = S^* + T^*$,
(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α),
(c) $(AT)^* = T^* A^*$, and
(d) $\|T^*\| = \|T\|$.

Proof of (a). Let $S, T \in \mathcal{B}(X, y)$ and $f \in Y^*$.

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Proof of (a). Let $S, T \in \mathcal{B}(X, y)$ and $f \in Y^*$. Then for all $x \in X$

$$\begin{aligned} ((S+T)^* f)x &= f((S + Tx)) \text{ by the definition of adjoint} \\ &= f(Sx + Tx)\text{by definition of operator addition} \\ &= f(Sx) + f(Tx) \text{ since } f \text{ is linear} \\ &= (S^* f)(x) + (T^* f)(x) \text{ by definition of adjoint.} \end{aligned}$$

So $(S + T)^* = S^* + T^*$. $\square$

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Proof of (a). Let $S, T \in \mathcal{B}(X, y)$ and $f \in Y^*$. Then for all $x \in X$

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So $(S + T)^* = S^* + T^*$. □

Theorem 6.6(b)

Theorem 6.6. Properties of the Adjoint in the Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α).

Proof of (b). Let $T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$, and $f \in Y^*$. Then for all $x \in X$ we have

$$\begin{aligned} ((\alpha T)^* f)(x) &= f((\alpha T)x) \text{ by the definition of adjoint} \\ &= f(\alpha(Tx)) \text{ by the definition of scalar multiplication} \\ &\qquad \text{on an operator} \\ &= \alpha f(Tx) \text{ since } f \text{ is linear} \\ &= \alpha(T^* f)(x) \text{ by the definition of adjoint.} \end{aligned}$$

So $(\alpha T)^* = \alpha T^*$. □

Theorem 6.6(b)

Theorem 6.6. Properties of the Adjoint in the Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α).

Proof of (b). Let $T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$, and $f \in Y^*$. Then for all $x \in X$ we have

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So $(\alpha T)^* = \alpha T^*$. □

Theorem 6.6(c)

Theorem 6.6. Properties of Adjoint in Normed Linear Space Setting. For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(c) $(AT)^* = T^*A^*$.

Proof of (c). Let $A \in \mathcal{B}(Y, Z)$, $T \in \mathcal{B}(X, Y)$, and $f \in Z^*$. We need to show that $(AT)^*f = T^*(A^*f)$. For all $x \in X$ we have

$$\begin{aligned}
((AT)^*f)(x) &= f((AT)(x)) \text{ by the definition of adjoint } (AT)^* \\
&\quad\ (\text{notice that } (AT)(x) \in Z \text{ and } f \in Z^*) \\
&= f(A(T(x))) \\
&= (A^*f)(T(x)) \text{ by the definition of adjoint } A^* \\
&\quad\ (\text{notice } T(x) \in Y \text{ and } Af \in Y^*) \\
&= (T^*(A^*f))(x) \text{ by definition of adjoint } T^* \text{ in terms of } A^*f \\
&\quad\ (\text{notice } T^*(A^*f) \in X^* \text{ and so } T^*(A^*f) : X \to \mathbb{F}) \\
&= ((T^*A^*)f)(x) \text{ since operator composition is associative.}
\end{aligned}$$

So $(AT)^* = T^*A^*$. $\square$

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Theorem 6.6(d)

Theorem 6.6. Properties of Adjoint in Normed Linear Space Setting. For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(d) $\|T^*\| = \|T\|$.

Proof of (d). We can scale to assume without loss of generality that $\|T\| = 1$. Then for any unit vectors $f \in Y^*$ and $x \in X$, we have

$$\begin{aligned} \|(T^*f)(x)\| &= |f(Tx)\| \text{ by the definition of adjoint } T^* \\ &\leq \|f\|\|Tx\| \text{ by definition of norm of functional } f \\ &\leq \|f\|\|T\|\|x\| \text{ by definition of } \|T\|, \text{ since } \|T\| = \|x\| = 1, \end{aligned}$$

so taking a supremum over all such $x \in X$, we have $\|T^*f\| \leq 1$, and taking a supremum over all such f, we have $\|T^*\| \leq 1$.

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so taking a supremum over all such $x \in X$, we have $\|T^*f\| \leq 1$, and taking a supremum over all such f, we have $\|T^*\| \leq 1$. Conversely, given any $\varepsilon > 0$, choose a unit vector $x \in X$ such that $\|Tx\| \geq 1 - \varepsilon$ (this can be done since $\|T\| = 1$).

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Proof of (d) continued. So

$$\begin{aligned} |f(Tx)| &= |(T^*f)(x)| \text{ by the definition of } T^* \\ &= \|Tx\| \geq 1 - \varepsilon. \end{aligned}$$

Taking a supremum over all unit vectors $x \in X$ gives $\|fT\| \geq 1 - \varepsilon$. Taking a supremum over all unit vectors $f \in Y^*$ implies that $\|ft\| = \|T^*\| \geq 1 - \varepsilon$.

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