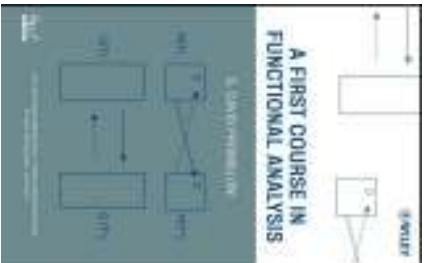


Introduction to Functional Analysis

Chapter 6. Duality

6.3. Double Duals and Reflexivity—Proofs of Theorems



Theorem 6.8

Theorem 6.8. The mapping $x \rightarrow \hat{x}$ (which maps X to X^{**}) is a linear isometry.

Proof. Let $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$. Then for all $f \in X^*$ we have that

$$\begin{aligned} \widehat{(x_1 + x_2)}f &= f(x_1 + x_2) \text{ by the definition of } \hat{} \\ &= f(x_1) + f(x_2) \text{ since } f \text{ is linear} \\ &= \hat{x}_1 + \hat{x}_2 \text{ by the definition of } \hat{}, \end{aligned}$$

and

$$\begin{aligned} \widehat{(\alpha x_1)}f &= f(\alpha x_1) \text{ by the definition of } \hat{} \\ &= \alpha f(x_1) \text{ since } f \text{ is linear} \\ &= \alpha \hat{x}_1 \text{ by the definition of } \hat{}. \end{aligned}$$

So the mapping is linear.

Theorem 6.8

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Proof (continued). Next,

$$\begin{aligned} \|x\| &= \sup\{|f(x)| \mid f \in X^*, \|f\| \leq 1\} \text{ by Corollary 5.7} \\ &= \sup\{|\hat{x}(f)| \mid f \in X^*, \|f\| \leq 1\} \text{ since } \hat{x}(f) = f(x) \\ &= \|\hat{x}\| \text{ by the definition of } \|\hat{x}\| \end{aligned}$$

□

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So the mapping is linear.

□

Theorem 6.9. General Uniform Boundedness Principle

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Let A be a subset of a normed linear space X such that for all $f \in X^*$ we have that $f(A)$ is a bounded set of scalars. Then A is bounded.

Proof. Let $A \subseteq X$ is a normed linear space X such that for all $f \in X^*$, we have $f(A) = \{f(a) \mid a \in A\}$ is bounded. Now for each $a \in A$, we have $f(a) = \hat{a}(f)$ where $\hat{a} \in X^{**}$. So for each $f \in X^*$,

$$f(A) = \{f(a) \mid a \in A\} = \{\hat{a}(f) \mid a \in A\}$$

is bounded. So $\{\hat{a}(f) \mid a \in A\}$ is a bounded set for each $f \in X^*$. Also, by Theorem 2.15, X^* is complete. So by the Uniform Boundedness Principle (Theorem 3.10), $\{\hat{a}(f) \mid a \in A\}$ is bounded in X^{**} for all $f \in X^*$. Since there is an isometry between X and X^{**} (which maps a to \hat{a}) then set A is bounded. □

Theorem 6.10

Theorem 6.10. L^p is reflexive for $1 < p < \infty$.

Proof. Let q satisfy the equation $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 6.3, $(L^p)^* = L^q$ and $(L^p)^{**} = (L^q)^* = L^p$. So L^p is isometric to $(L^p)^{**}$. We need to explicitly find a surjective isometry from L^p to $(L^p)^{**}$. Let U be the surjective isometry from L^q to $(L^p)^*$ given in Theorem 6.3. For $g \in L^q$ define $U(g) = \varphi_g \in (L^p)^*$ where $\varphi_g(f) = \int_E fg \, d\mu$ for all $f \in L^p$. Let V be the surjective isometry from L^p to $(L^q)^*$. Notice then that the adjoint U^* maps $(L^p)^{**}$ to $(L^q)^*$. Consider $(U^*)^{-1}V : L^p \rightarrow (L^p)^{**}$.

Theorem 6.10 (continued again)

Theorem 6.10. L^p is reflexive for $1 < p < \infty$.

Proof (continued again). Finally, by the Riesz-Representation Theorem for L^p (not explicitly stated in our text, but see Royden and Fitzpatrick's *Real Analysis* 4th Edition, Section 8.1) U and V are bijections (one to one and onto) and so U^* is bijective and $(U^*)^{-1}$ exists and is bijective. So $(U^*)^{-1}V$ is surjective (onto). Therefore, $(U^*)^{-1}V$ is the surjective isometry required. \square

Theorem 6.10 (continued)

Theorem 6.10. L^p is reflexive for $1 < p < \infty$.

Proof (continued). For all $f \in L^p$ and $g \in L^q$ we have

$$\begin{aligned} (U^*\hat{f})(g) &= \hat{f}(Ug) \text{ by the definition of adjoint } U^* \\ &= (Ug)f \text{ by definition of } \hat{f} \\ &= \int_E fg \, d\mu \text{ by definition of } U(g) = \varphi_g \\ &= \int_E gf \, d\mu \\ &= (Vf)(g) \text{ by definition of } V(f) = \varphi_f. \end{aligned}$$

So $(U^*\hat{f})(g) = (Vf)(g)$ for all $g \in L^q$ and hence $U^*\hat{f} = Vf$ (which are linear functionals from L^q to \mathbb{R} ; that is, elements of $(L^q)^*$). Therefore $(U^*)^{-1}U^*\hat{f} = (U^*)^{-1}Vf$ or $\hat{f} = (U^*)^{-1}Vf$ and $(U^*)^{-1}V$ maps f to \hat{f} . By Theorem 6.8, $(U^*)^{-1}V$ is a linear isometry from L^p (where f lives) to $(L^p)^{**}$ (where \hat{f} lives).

Theorem 6.11

Theorem 6.11. A closed subspace of a reflexive Banach space is reflexive.

Proof. Suppose Z is a closed subspace of reflexive Banach space X . Let $\varphi \in Z^{**}$. To show that Z is reflexive, we must find $x \in Z$ such that $\hat{x} = \varphi$ (then show that the embedding of Z into Z^{**} given by $x \mapsto \hat{x}$ is surjective [onto]). For $f \in X^*$, denote the restriction of f to Z as $f|Z$. Define $\eta \in X^{**}$ as $\eta(f) = \varphi(f|Z)$ for all $f \in X^*$. Since the norm of $f|Z$ is less than or equal to the norm of f (since the sup is taken over the smaller set Z in determining $\|f|Z\|$),

$$|\eta(f)| = |\varphi(f|Z)| \leq \|\varphi\| \|f|Z\| \leq \|\varphi\| \|f\| \text{ for all } f \in X^*,$$

and so $\|\eta\| = \sup\{|\eta(f)| \mid \|f\| = 1\} \leq \|\varphi\|$. Since X is reflexive, there is $x \in X$ such that $\hat{x} = \eta$. Given any $f \in X^*$ with $f(Z) = 0$ (i.e., functional f is 0 on subspace Z), we have $\hat{x}(f) = \eta(f) = \varphi(f|Z) = 0$. So $\hat{x}(f) = f(x) = 0$ for all $f \in X^*$.

Theorem 6.11 (continued)

Proof (continued). By Corollary 5.5, since Z is a closed subspace of X , if $x \notin Z$ then there would be $f \in X^*$ for which $f(Z) = 0$ and $f(x) = 1$. Since this is not the case, it must be that $x \in Z$. We need to show that $\hat{x} = \varphi$ (where the hat indicated embedding Z in Z^{**} , whereas above the hat indicated embedding X in X^{**}). Let $g \in Z^*$ be arbitrary. By the Normed Linear Space Version of the Hahn-Banach Theorem (Theorem 5.4), there is an extension of g to all of X , say $f \in X^*$. Then

$$\begin{aligned} \varphi(f) &= \varphi(f|Z) \text{ since } f \text{ extends } g \text{ from } Z \text{ to } X \\ &= \eta(f) \text{ by the definition of } \eta \\ &= \hat{x}(f) \text{ by the choice of } x \text{ above} \\ &= f(x) \text{ by definition of } \hat{\cdot} \text{ in the } X \text{ embedding} \\ &= g(x) \text{ since } x \in Z \text{ as shown above and } f \text{ extends } g \\ &= \hat{x}(g) \text{ be the definition of } \hat{\cdot} \text{ in the } Z \text{ embedding.} \end{aligned}$$

So, $x \in Z$ and $\hat{x} = \varphi$, as desired. \square

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Theorem 6.12 (continued)

Theorem 6.12. A Banach space X is reflexive if and only if X^* is reflexive.

Proof (continued). So $\Delta = \hat{g}$, the mapping $g \mapsto \hat{g}$ is surjective, and X^* is reflexive.

Conversely, if X^* is reflexive then by the above argument X^{**} is reflexive. So X is a subspace of Banach space X^{**} (remember that X^* and X^{**} are complete by Theorem 2.15), then by Theorem 2.16, X is a closed subspace of reflexive space X^{**} , and by Theorem 6.11, X is reflexive. \square

Theorem 6.12

Theorem 6.12. A Banach space X is reflexive if and only if X^* is reflexive.

Proof. Suppose X is reflexive. For any $\Delta \in X^{***}$, define $g \in X^*$ as $g(x) = \Delta(\hat{x})$ (where $x \in X$ and $\hat{x} \in X^{**}$ where the correspondence of x and \hat{x} is as given at the beginning of this section). We claim that $\hat{g} = \Delta$ (which would show that for any $\Delta \in X^{***}$, there is $g \in X^*$ such that $g \mapsto \hat{g} = \Delta$, and so the mapping is surjective [onto] and X^* is reflexive). Let $\varphi \in X^{**}$ be arbitrary. There is $x \in X$ with $\hat{x} = \varphi$ since X is reflexive. Then

$$\begin{aligned} \Delta(\varphi) &= \Delta(\hat{x}) \text{ since } \hat{x} = \varphi \\ &= g(x) \text{ by the definition of } g \\ &= \hat{x}(g) \text{ by definition of } \hat{\cdot} \text{ in the } X \text{ embedding} \\ &= \varphi(g) \text{ since } \hat{x} = \varphi \\ &= \hat{g}(\varphi) \text{ by the definition of } \hat{\cdot} \text{ in the } X^* \text{ embedding.} \end{aligned}$$

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