Introduction to Functional Analysis

Chapter 6. Duality

6.3. Double Duals and Reflexivity—Proofs of Theorems

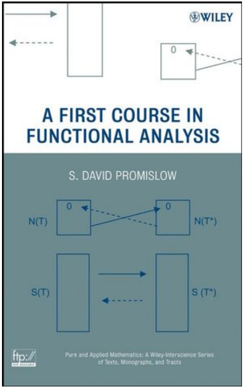

Table of contents

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$$\begin{aligned} \|x\| &= \sup\{|f(x)| \mid f \in X^*, \|f\| \leq 1\} \text{ by Corollary 5.7} \\ &= \sup\{|\hat{x}(f)| \mid f \in X^*, \|f\| \leq 1\} \text{ since } \hat{x}(f) = f(x) \\ &= \|\hat{x}\| \text{ by the definition of } \|\hat{x}\| \end{aligned}$$

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Let A be a subset of a normed linear space X such that for all $f \in X^*$ we have that $f(A)$ is a bounded set of scalars. Then A is bounded.

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$$\begin{aligned}(U^*\hat{f})(g) &= \hat{f}(Ug) \text{ by the definition of adjoint } U^* \\ &= (Ug)f \text{ by definition of } \hat{f} \\ &= \int_E fg\,d\mu \text{ by definition of } U(g) = \varphi_g \\ &= \int_E gf\,d\mu \\ &= (Vf)(g) \text{ by definition of } V(f) = \varphi_f.\end{aligned}$$

So $(U^*\hat{f})(g) = (Vf)(g)$ for all $g \in L^q$ and hence $U^*\hat{f} = Vf$ (which are linear functionals from L^q to $\mathbb{F}$; that is, elements of $(L^q)^*$).

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Proof (continued again). Finally, by the Riesz-Representation Theorem for L^p (not explicitly stated in our text, but see Royden and Fitzpatrick's *Real Analysis* 4th Edition, Section 8.1) U and V are bijections (one to one and onto) and so U^* is bijective and $(U^*)^{-1}$ exists and is bijective. So $(U^*)^{-1}V$ is surjective (onto). Therefore, $(U^*)^{-1}V$ is the surjective isometry required. □

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$$|\eta(f)| = |\varphi(f|Z)| \leq \|\varphi\|\|f|Z\| \leq \|\varphi\|\|f\| \text{ for all } f \in X^*,$$

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Theorem 6.11 (continued)

Proof (continued). By Corollary 5.5, since Z is a closed subspace of X, if $x \notin Z$ then there would be $f \in X^*$ for which $f(Z) = 0$ and $f(x) = 1$. Since this is not the case, it must be that $x \in Z$. We need to show that $\hat{x} = \varphi$ (where the hat indicated embedding Z in Z^{**}, whereas above the hat indicated embedding X in X^{**}). Let $g \in Z^*$ be arbitrary.

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$$\begin{aligned}
\varphi(f) &= \varphi(f|Z) \text{ since } f \text{ extends } g \text{ from } Z \text{ to } Xv \\
&= \eta(f) \text{ by the definition of } \eta \\
&= \hat{x}(f) \text{ by the choice of } x \text{ above} \\
&= f(x) \text{ by definition of } \hat{} \text{ in the } X \text{ embedding} \\
&= g(x) \text{ since } x \in Z \text{ as shown above and } f \text{ extends } g \\
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So, $x \in Z$ and $\hat{x} = \varphi$, as desired. □

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$$\begin{aligned}
\Delta(\varphi) &= \Delta(\hat{x}) \text{ since } \hat{x} = \varphi \\
&= g(x) \text{ by the definition of } g \\
&= \hat{x}(g) \text{ by definition of } \hat{} \text{ in the } X \text{ embedding} \\
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$$\begin{aligned}
\Delta(\varphi) &= \Delta(\hat{x}) \text{ since } \hat{x} = \varphi \\
&= g(x) \text{ by the definition of } g \\
&= \hat{x}(g) \text{ by definition of } \hat{\ } \text{ in the } X \text{ embedding} \\
&= \varphi(g) \text{ since } \hat{x} = \varphi \\
&= \hat{g}(\varphi) \text{ by the definition of } \hat{\ } \text{ in the } X^* \text{ embedding.}
\end{aligned}$$

Theorem 6.12 (continued)

Theorem 6.12. A Banach space X is reflexive if and only if X^* is reflexive.

Proof (continued). So $\Delta = \hat{g}$, the mapping $g \mapsto \hat{g}$ is surjective, and X^* is reflexive.

Conversely, if X^* is reflexive then by the above argument X^{**} is reflexive.

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Conversely, if X^* is reflexive then by the above argument X^{**} is reflexive. So X is a subspace of Banach space X^{**} (remember that X^* and X^{**} are complete by Theorem 2.15), then by Theorem 2.16, X is a closed subspace of reflexive space X^{**}, and by Theorem 6.11, X is reflexive. $\square$

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