Chapter 6. Duality
6.3. Double Duals and Reflexivity—Proofs of Theorems
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Theorem 6.8

Theorem 6.8. The mapping $x \mapsto \hat{x}$ (which maps $X$ to $X^{**}$) is a linear isometry.

Proof. Let $x_1, x_2 \in X$ and $\alpha \in F$. 


**Theorem 6.8.** The mapping \( x \rightarrow \hat{x} \) (which maps \( X \) to \( X^{**} \)) is a linear isometry.

**Proof.** Let \( x_1, x_2 \in X \) and \( \alpha \in \mathbb{F} \). Then for all \( f \in X^* \) we have that

\[
(x_1 + x_2)f = f(x_1 + x_2) \quad \text{by the definition of } \hat{x} \\
= f(x_1) + f(x_2) \quad \text{since } f \text{ is linear} \\
= \hat{x}_1 + \hat{x}_2 \quad \text{by the definition of } \hat{x},
\]
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Proof. Let $x_1, x_2 \in X$ and $\alpha \in \mathbb{F}$. Then for all $f \in X^*$ we have that

$$\hat{(x_1 + x_2)}f = f(x_1 + x_2) \text{ by the definition of } \hat{\cdot}$$
$$= f(x_1) + f(x_2) \text{ since } g \text{ is linear}$$
$$= \hat{x}_1 + \hat{x}_2 \text{ by the definition of } \hat{\cdot},$$

and

$$\hat{(\alpha x_1)}f = f(\alpha x_1) \text{ by the definition of } \hat{\cdot}$$
$$= \alpha f(x_1) \text{ since } f \text{ is linear}$$
$$= \alpha \hat{x}_1 \text{ by the definition of } \hat{\cdot}.$$

So the mapping is linear.
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So the mapping is linear.
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Proof (continued). Next,

\[
\|x\| = \sup\{\|f(x)\| \mid f \in X^*, \|f\| \leq 1\} \text{ by Corollary 5.7}
\]
\[
= \sup\{\|\hat{x}(f)\| \mid f \in X^*, \|f\| \leq 1\} \text{ since } \hat{x}(f) = f(x)
\]
\[
= \|\hat{x}\| \text{ by the definition of } \|\hat{x}\|
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Theorem 6.8. The mapping $x \rightarrow \hat{x}$ (which maps $X$ to $X^{**}$) is a linear isometry.

Proof (continued). Next,

$$||x|| = \sup\{||f(x)|| \mid f \in X^*, ||f|| \leq 1\} \text{ by Corollary 5.7}$$

$$= \sup\{||\hat{x}(f)|| \mid f \in X^*, ||f|| \leq 1\} \text{ since } \hat{x}(f) = f(x)$$

$$= ||\hat{x}|| \text{ by the definition of } ||\hat{x}||$$
Theorem 6.9. General Uniform Boundedness Principle

Let $A$ be a subset of a normed linear space $X$ such that for all $f \in X^*$ we have that $f(A)$ is a bounded set of scalars. Then $A$ is bounded.

Proof. Let $A \subseteq X$ is a normed linear space $X$ such that for all $f \in X^*$, we have $f(A) = \{f(a) \mid a \in A\}$ is bounded.

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Proof. Let $A \subseteq X$ is a normed linear space $X$ such that for all $f \in X^*$, we have $f(A) = \{f(a) \mid a \in A\}$ is bounded. Now for each $a \in A$, we have $f(a) = \hat{a}(f)$ where $\hat{a} \in X^{**}$.
Let \( A \) be a subset of a normed linear space \( X \) such that for all \( f \in X^* \) we have that \( f(A) \) is a bounded set of scalars. Then \( A \) is bounded.

Proof. Let \( A \subseteq X \) is a normed linear space \( X \) such that for all \( f \in X^* \), we have \( f(A) = \{ f(a) \mid a \in A \} \) is bounded. Now for each \( a \in A \), we have \( f(a) = \hat{a}(f) \) where \( \hat{a} \in X^{**} \). So for each \( f \in X^* \),

\[
f(A) = \{ f(a) \mid a \in A \} = \{ \hat{a}(f) \mid a \in A \}
\]

is bounded. So \( \{ \hat{a}(f) \mid a \in A \} \) is a bounded set for each \( f \in X^* \).
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**Proof.** Let $A \subseteq X$ is a normed linear space $X$ such that for all $f \in X^*$, we have $f(A) = \{f(a) | a \in A\}$ is bounded. Now for each $a \in A$, we have $f(a) = \hat{a}(f)$ where $\hat{a} \in X^{**}$. So for each $f \in X^*$,

$$f(A) = \{f(a) | a \in A\} = \{\hat{a}(f) | a \in A\}$$

is bounded. So $\{\hat{a}(f) | a \in A\}$ is a bounded set for each $f \in X^*$. Also, by Theorem 2.15, $X^*$ is complete. So by the Uniform Boundedness Principle (Theorem 3.10), $\{\hat{a}(f) | a \in A\}$ is bounded in $X^{**}$ for all $f \in X^*$. 
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is bounded. So $\{\hat{a}(f) \mid a \in A\}$ is a bounded set for each $f \in X^*$. Also, by Theorem 2.15, $X^*$ is complete. So by the Uniform Boundedness Principle (Theorem 3.10), $\{\hat{a}(f) \mid a \in A\}$ is bounded in $X^{**}$ for all $f \in X^*$. Since there is an isometry between $X$ and $X^{**}$ (which maps $a$ to $\hat{a}$) then set $A$ is bounded. \qed

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**Proof.** Let \( A \subseteq X \) is a normed linear space \( X \) such that for all \( f \in X^* \), we have \( f(A) = \{ f(a) \mid a \in A \} \) is bounded. Now for each \( a \in A \), we have \( f(a) = \hat{a}(f) \) where \( \hat{a} \in X^{**} \). So for each \( f \in X^* \),

\[
f(A) = \{ f(a) \mid a \in A \} = \{ \hat{a}(f) \mid a \in A \}
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is bounded. So \( \{ \hat{a}(f) \mid a \in A \} \) is a bounded set for each \( f \in X^* \). Also, by Theorem 2.15, \( X^* \) is complete. So by the Uniform Boundedness Principle (Theorem 3.10), \( \{ \hat{a}(f) \mid a \in A \} \) is bounded in \( X^{**} \) for all \( f \in X^* \). Since there is an isometry between \( X \) and \( X^{**} \) (which maps \( a \) to \( \hat{a} \)) then set \( A \) is bounded. \(
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Theorem 6.10. \( L^p \) is reflexive for \( 1 < p < \infty \).

Proof. Let \( q \) satisfy the equation \( \frac{1}{p} + \frac{1}{q} = 1 \).
Theorem 6.10. \( L^p \) is reflexive for \( 1 < p < \infty \).

**Proof.** Let \( q \) satisfy the equation \( \frac{1}{p} + \frac{1}{q} = 1 \). By Theorem 6.3, \( (L^p)^* = L^q \) and \( (L^p)^{**} = (L^q)^* = L^p \). So \( L^p \) is isometric to \( (L^p)^{**} \). We need to explicitly find a surjective isometry from \( L^p \) to \( (L^p)^{**} \).
Theorem 6.10. $L^p$ is reflexive for $1 < p < \infty$.

Proof. Let $q$ satisfy the equation $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 6.3, $(L^p)^* = L^q$ and $(L^p)^{**} = (L^q)^* = L^p$. So $L^p$ is isometric to $(L^p)^{**}$. We need to explicitly find a surjective isometry from $L^p$ to $(L^p)^{**}$. Let $U$ be the surjective isometry from $L^q$ to $(L^p)^*$ given in Theorem 6.3: For $g \in L^q$ define $U(g) = \varphi_g \in (L^p)^*$ where $\varphi_g(f) = \int_E fg \, d\mu$ for all $f \in L^p$. Let $V$ be the surjective isometry from $L^p$ to $(L^q)^*$.
Theorem 6.10. \( L^p \) is reflexive for \( 1 < p < \infty \).

**Proof.** Let \( q \) satisfy the equation \( \frac{1}{p} + \frac{1}{q} = 1 \). By Theorem 6.3, \( (L^p)^* = L^q \) and \( (L^p)^{**} = (L^q)^* = L^p \). So \( L^p \) is isometric to \( (L^p)^{**} \). We need to explicitly find a surjective isometry from \( L^p \) to \( (L^p)^{**} \). Let \( U \) be the surjective isometry from \( L^q \) to \( (L^p)^* \) given in Theorem 6.3: For \( g \in L^q \) define \( U(g) = \varphi_g \in (L^p)^* \) where \( \varphi_g(f) = \int_E fg \, d\mu \) for all \( f \in L^p \). Let \( V \) be the surjective isometry from \( L^p \) to \( (L^q)^* \). Notice then that the adjoint \( U^* \) maps \( (L^p)^{**} \) to \( (L^q)^* \). Consider \( (U^*)^{-1}V : L^p \to (L^p)^{**} \).
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Theorem 6.10. $L^p$ is reflexive for $1 < p < \infty$.

Proof (continued). For all $f \in L^p$ and $g \in L^q$ we have

\[
(U^* \hat{f})(g) = \hat{f}(Ug) \text{ by the definition of adjoint } U^*
\]
\[
= (Ug)f \text{ by definition of } \hat{f}
\]
\[
= \int_E fg \, d\mu \text{ by definition of } U(g) = \varphi_g
\]
\[
= \int_E gf \, d\mu
\]
\[
= (Vf)(g) \text{ by definition of } V(f) = \varphi_f.
\]

So $(U^* \hat{f})(g) = (Vf)(g)$ for all $g \in L^q$ and hence $U^* \hat{f} = Vf$ (which are linear functionals from $L^q$ to $\mathbb{F}$; that is, elements of $(L^q)^*$).
Theorem 6.10 (continued)

Theorem 6.10. \( L^p \) is reflexive for \( 1 < p < \infty \).

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So \((U^*\hat{f})(g) = (Vf)(g)\) for all \( g \in L^q \) and hence \( U^*\hat{f} = Vf \) (which are linear functionals from \( L^q \) to \( F \); that is, elements of \((L^q)^*\)). Therefore \((U^*)^{-1}U^*\hat{f} = (U^*)^{-1}Vf\) or \( \hat{f} = (U^*)^{-1}Vf \) and \((U^*)^{-1}V\) maps \( f \) to \( \hat{f} \). By Theorem 6.8, \((U^*)^{-1}V\) is a linear isometry from \( L^p \) (where \( f \) lives) to \((L^p)^{**}\) (where \( \hat{f} \) lives).
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= (Vf)(g) \text{ by definition of } V(f) = \varphi_f.
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So \( (U^* \hat{f})(g) = (Vf)(g) \) for all \( g \in L^q \) and hence \( U^* \hat{f} = Vf \) (which are linear functionals from \( L^q \) to \( \mathbb{F} \); that is, elements of \( (L^q)^* \)). Therefore \( (U^*)^{-1} U^* \hat{f} = (U^*)^{-1} Vf \) or \( \hat{f} = (U^*)^{-1} Vf \) and \( (U^*)^{-1} V \) maps \( f \) to \( \hat{f} \). By Theorem 6.8, \( (U^*)^{-1} V \) is a linear isometry from \( L^p \) (where \( f \) lives) to \( (L^p)^{**} \) (where \( \hat{f} \) lives).
Theorem 6.10. \( L^p \) is reflexive for \( 1 < p < \infty \).

Proof (continued again). Finally, by the Riesz-Representation Theorem for \( L^p \) (not explicitly stated in our text, but see Royden and Fitzpatrick’s *Real Analysis* 4th Edition, Section 8.1) \( U \) and \( V \) are bijections (one to one and onto) and so \( U^* \) is bijective and \( (U^*)^{-1} \) exists and is bijective. So \( (U^*)^{-1}V \) is surjective (onto). Therefore, \( (U^*)^{-1}V \) is the surjective isometry required.
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Theorem 6.11

Theorem 6.11. A closed subspace of a reflexive Banach space is reflexive.

Proof. Suppose $Z$ is a closed subspace of reflexive Banach space $X$. Let $\varphi \in Z^{**}$. 

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**Proof.** Suppose $Z$ is a closed subspace of reflexive Banach space $X$. Let $\varphi \in Z^{**}$. To show that $Z$ is reflexive, we must find $x \in Z$ such that $\hat{x} = \varphi$ (then show that the embedding of $Z$ into $Z^{**}$ given by $x \mapsto \hat{x}$ is surjective [onto]).
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For \( f \in X^* \), denote the restriction of \( f \) to \( Z \) as \( f \mid Z \). Define \( \eta \in X^{**} \) as \( \eta(f) = \varphi(f \mid Z) \) for all \( f \in X^* \).
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For $f \in X^*$, denote the restriction of $f$ to $Z$ as $f|Z$. Define $\eta \in X^{**}$ as $\eta(f) = \varphi(f|Z)$ for all $f \in X^*$. Since the norm of $f|Z$ is less than or equal to the norm of $f$ (since the sup is taken over the smaller set $Z$ in determining $\|f|Z\|$),

$$|\eta(f)| = |\varphi(f|Z)| \leq \|\varphi\| \|f|Z\| \leq \|\varphi\| \|f\| \text{ for all } f \in X^*,$$

and so $\|\eta\| = \sup\{|\eta(f)| \mid \|f\| = 1\} \leq \|\varphi\|$. 


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For $f \in X^*$, denote the restriction of $f$ to $Z$ as $f | Z$. Define $\eta \in X^{**}$ as $\eta(f) = \varphi(f | Z)$ for all $f \in X^*$. Since the norm of $f | Z$ is less than or equal to the norm of $f$ (since the sup is taken over the smaller set $Z$ in determining $\|f | Z\|$),

$$|\eta(f)| = |\varphi(f | Z)| \leq \|\varphi\| \|f | Z\| \leq \|\varphi\| \|f\|$$

for all $f \in X^*$, and so $\|\eta\| = \sup\{|\eta(f)| \mid \|f\| = 1\} \leq \|\varphi\|$. Since $X$ is reflexive, there is $x \in X$ such that $\hat{x} = \eta$. Given any $f \in X^*$ with $f(Z) = 0$ (i.e., functional $f$ is 0 on subspace $Z$), we have $\hat{x}(f) = \eta(f) = \varphi(f | Z) = 0$. So $\hat{x}(f) = f(x) = 0$ for all $f \in X^*$.
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For $f \in X^*$, denote the restriction of $f$ to $Z$ as $f|Z$. Define $\eta \in X^{**}$ as $\eta(f) = \varphi(f|Z)$ for all $f \in X^*$. Since the norm of $f|Z$ is less than or equal to the norm of $f$ (since the sup is taken over the smaller set $Z$ in determining $\|f|Z\|$),

$$|\eta(f)| = |\varphi(f|Z)| \leq \|\varphi\| \|f|Z\| \leq \|\varphi\| \|f\| \text{ for all } f \in X^*,$$

and so $\|\eta\| = \sup\{|\eta(f)| \mid \|f\| = 1\} \leq \|\varphi\|$. Since $X$ is reflexive, there is $x \in X$ such that $\hat{x} = \eta$. Given any $f \in X^*$ with $f(Z) = 0$ (i.e., functional $f$ is 0 on subspace $Z$), we have $\hat{x}(f) = \eta(f) = \varphi(f|Z) = 0$. So

$\hat{x}(f) = f(x) = 0$ for all $f \in X^*$. 
Theorem 6.11 (continued)

**Proof (continued).** By Corollary 5.5, since $Z$ is a closed subspace of $X$, if $x \notin Z$ then there would be $f \in X^*$ for which $f(Z) = 0$ and $f(x) = 1$. Since this is not the case, it must be that $x \in Z$. We need to show that $\hat{x} = \varphi$ (where the hat indicated embedding $Z$ in $Z^{**}$, whereas above the hat indicated embedding $X$ in $X^{**}$). Let $g \in Z^*$ be arbitrary.
Theorem 6.11 (continued)

**Proof (continued).** By Corollary 5.5, since $Z$ is a closed subspace of $X$, if $x \notin Z$ then there would be $f \in X^*$ for which $f(Z) = 0$ and $f(x) = 1$. Since this is not the case, it must be that $x \in Z$. We need to show that $\hat{x} = \varphi$ (where the hat indicated embedding $Z$ in $Z^{**}$, whereas above the hat indicated embedding $X$ in $X^{**}$). Let $g \in Z^*$ be arbitrary. By the Normed Linear Space Version of the Hahn-Banach Theorem (Theorem 5.4), there is an extension of $g$ to all of $X$, say $f \in X^*$. Then

$$
\varphi(f) = \varphi(f|Z) \text{ since } f \text{ extends } g \text{ from } Z \text{ to } X^*,
$$

$$
= \eta(f) \text{ by the definition of } \eta,
$$

$$
= \hat{x}(f) \text{ by the choice of } x \text{ above},
$$

$$
= f(x) \text{ by definition of } \hat{\cdot} \text{ in the } X \text{ embedding},
$$

$$
= g(x) \text{ since } x \in Z \text{ as shown above and } f \text{ extends } g,
$$

$$
= \hat{x}(g) \text{ be the definition of } \hat{\cdot} \text{ in the } Z \text{ embedding}.
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So, $x \in Z$ and $\hat{x} = \varphi$, as desired. \qed
Theorem 6.11 (continued)

Proof (continued). By Corollary 5.5, since $Z$ is a closed subspace of $X$, if $x \not\in Z$ then there would be $f \in X^*$ for which $f(Z) = 0$ and $f(x) = 1$. Since this is not the case, it must be that $x \in Z$. We need to show that $\hat{x} = \varphi$ (where the hat indicated embedding $Z$ in $Z^{**}$, whereas above the hat indicated embedding $X$ in $X^{**}$). Let $g \in Z^*$ be arbitrary. By the Normed Linear Space Version of the Hahn-Banach Theorem (Theorem 5.4), there is an extension of $g$ to all of $X$, say $f \in X^*$. Then

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\varphi(f) = \varphi(f|Z) \text{ since } f \text{ extends } g \text{ from } Z \text{ to } X
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$$

So, $x \in Z$ and $\hat{x} = \varphi$, as desired.  \qed
Theorem 6.12

**Theorem 6.12.** A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

**Proof.** Suppose $X$ is reflexive.
Theorem 6.12

**Theorem 6.12.** A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

**Proof.** Suppose $X$ is reflexive. For any $\Delta \in X^{***}$, define $g \in X^*$ as $g(x) = \Delta(\hat{x})$ (where $x \in X$ and $\hat{x} \in X^{**}$ where the correspondence of $x$ and $\hat{x}$ is as given at the beginning of this section).
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Proof. Suppose $X$ is reflexive. For any $\Delta \in X^{***}$, define $g \in X^*$ as $g(x) = \Delta(\hat{x})$ (where $x \in X$ and $\hat{x} \in X^{**}$ where the correspondence of $x$ and $\hat{x}$ is as given at the beginning of this section). We claim that $\hat{g} = \Delta$ (which would show that for any $\Delta \in X^{***}$, there is $g \in X^*$ such that $g \mapsto \hat{g} = \Delta$, and so the mapping is surjective [onto] and $X^*$ is reflexive). Let $\varphi \in X^{**}$ be arbitrary.
Theorem 6.12

Theorem 6.12. A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

Proof. Suppose $X$ is reflexive. For any $\Delta \in X^{***}$, define $g \in X^*$ as $g(x) = \Delta(\hat{x})$ (where $x \in X$ and $\hat{x} \in X^{**}$ where the correspondence of $x$ and $\hat{x}$ is as given at the beginning of this section). We claim that $\hat{g} = \Delta$ (which would show that for any $\Delta \in X^{***}$, there is $g \in X^*$ such that $g \mapsto \hat{g} = \Delta$, and so the mapping is surjective [onto] and $X^*$ is reflexive). Let $\varphi \in X^{**}$ be arbitrary. There is $x \in X$ with $\hat{x} = \varphi$ since $X$ is reflexive. Then

$$\Delta(\varphi) = \Delta(\hat{x}) \text{ since } \hat{x} = \varphi$$
$$= g(x) \text{ by the definition of } g$$
$$= \hat{x}(g) \text{ by definition of } \hat{ } \text{ in the } X \text{ embedding}$$
$$= \varphi(g) \text{ since } \hat{x} = \varphi$$
$$= \hat{g}(\varphi) \text{ by the definition of } \hat{ } \text{ in the } X^* \text{ embedding}.$$
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Let $\varphi \in X^{**}$ be arbitrary. There is $x \in X$ with $\hat{x} = \varphi$ since $X$ is reflexive. Then

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\]
Theorem 6.12 (continued)

**Theorem 6.12.** A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

**Proof (continued).** So $\Delta = \hat{g}$, the mapping $g \mapsto \hat{g}$ is surjective, and $X^*$ is reflexive.

Conversely, if $X^*$ is reflexive then by the above argument $X^{**}$ is reflexive.
Theorem 6.12. A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

**Proof (continued).** So $\Delta = \hat{g}$, the mapping $g \mapsto \hat{g}$ is surjective, and $X^*$ is reflexive.

Conversely, if $X^*$ is reflexive then by the above argument $X^{**}$ is reflexive. So $X$ is a subspace of Banach space $X^{**}$ (remember that $X^*$ and $X^{**}$ are complete by Theorem 2.15), then by Theorem 2.16, $X$ is a closed subspace of reflexive space $X^{**}$, and by Theorem 6.11, $X$ is reflexive. $\square$
Theorem 6.12. A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.

Proof (continued). So $\Delta = \hat{g}$, the mapping $g \mapsto \hat{g}$ is surjective, and $X^*$ is reflexive. Conversely, if $X^*$ is reflexive then by the above argument $X^{**}$ is reflexive. So $X$ is a subspace of Banach space $X^{**}$ (remember that $X^*$ and $X^{**}$ are complete by Theorem 2.15), then by Theorem 2.16, $X$ is a closed subspace of reflexive space $X^{**}$, and by Theorem 6.11, $X$ is reflexive. \qed