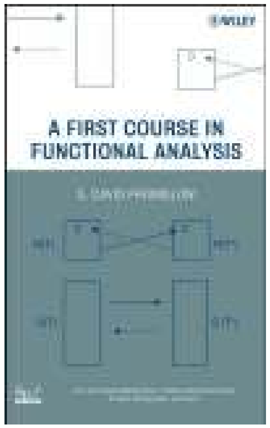
Introduction to Functional Analysis

Chapter 6. Duality

6.4. Weak and Weak* Convergence—Proofs of Theorems



Lemma

Lemma. If (x_n) is convergent to x in X then (x_n) is weakly convergent to x.

Proof. Suppose $(x_n) \to x$. Let $\varepsilon > 0$ and let $f \in X^*$ with $f \neq 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|x - x_n\| < \varepsilon/\|f\|$. So if $f \in X^*$ then for $n \geq N$ we have

$$\|f(x) - f(x_n)\| = \|f(x - x_n)\| \leq \|f\|\|x - x_n\| < \|f\|\varepsilon/\|f\| = \varepsilon.$$

So $f(x_n) \to f(x)$ and (x_n) is weakly convergent to x. □

Proposition 6.15, Continuity of Operations.

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For any sequence (x_n) which converges weakly to x, any sequence (y_n) which converges weakly to y, and any sequence of scalars (α_n) converging to α, we have:

(a) $(x_n + y_n)$ converges weakly to $x + y$,

(b) $(\alpha_n x_n)$ converges weakly to αx.

Proof of (a). Let $f \in X^*$ be arbitrary. Then we have $f(x_n) \to f(x)$ and $f(y_n) \to f(y)$. So $f(x_n + y_n) = f(x_n) + f(y_x) \to x + y$. Since $f \in X^*$ is arbitrary, $(x_n + y_n)$ converges weakly to $x + y$. □

Proposition 6.15(b).

Proposition 6.15. Continuity of Operations.
For any sequence (x_n) which converges weakly to x, any sequence (y_n) which converges weakly to y, and any sequence of scalars (α_n) converging to α, we have:

(a) $(x_n + y_n)$ converges weakly to $x + y$,

(b) $(\alpha_n x_n)$ converges weakly to αx.

Proof of (b). Let $f \in X^*$ be arbitrary. Then we have $(\alpha_n) \to \alpha$ and $(f(x_n)) \to x$. Notice that (α_n) and $f(x_n))$ are both sequences in $\mathbb{F}$. The limit of the product of two convergent sequences is the product of the limits, so $(f(\alpha_n x_n)) = (\alpha_n f(x_n)) \to \alpha f(x) = f(\alpha x)$. Since $f \in X^*$ is arbitrary, $(\alpha_n x_n)$ converges weakly to αx. □

Proposition 6.16

Proposition 6.16. If a sequence (x_n) in ℓ^1 converges weakly to x, then (x_n) converges to x with respect to the ℓ^1 norm. We take $\mathbb{F} = \mathbb{C}$.

Proof. By the linearity of $f \in X^*$, without loss of generality we may assume $x = 0$. Suppose (x_n) converges weakly to $x = 0$. ASSUME (x_n) does not converge to $x = 0$ with respect to the ℓ^1 norm. Then (x_n) has a subsequence $x_m)$ such that there is an $r > 0$ where $\|x_m\}_1 \geq 5r$ for all $m \in \mathbb{N}$ (some terms of (x_n) could be close to 0, but infinitely many are not).
Let $k, c \in \mathbb{N}$. Choose $n > k$ large enough so that $\|x_m(i)\|_1 \leq r/(c-1)$ for $i = 1, 2, \ldots, c-1$ (here, $x_m(i)$ represents the ith term of x_m). This can be done since we view only the first $c-1$ coordinates of x_m and hence this is a claim about a finite dimensional space, namely $\mathbb{F}^{c-1}$. In finite dimensions, weak convergence implies convergence (Exercise 6.9).

Proposition 6.16, Part 2

Proposition 6.16. If a sequence (x_n) in ℓ^1 converges weakly to x, then (x_n) converges to x with respect to the ℓ^1 norm. We take $\mathbb{F} = \mathbb{C}$.

Proof (Part 2). So in $\mathbb{F}^{c-1}$, the "pieces" of x_n converge to 0 in terms of the ℓ^1 norm. Next, choose $d \in \mathbb{N}$ large enough so that $\sum_{i=d+1}^{\infty} |x_m(i)| < r$ (this can be done since $x_m \in \ell^1$). It follows that

$$\begin{aligned}\sum_{i=c}^{d} |x_m(i)| &= \|x_m\|_1 = \sum_{i=1}^{c-1} |x_m(i)| - \sum_{i=d+1}^{\infty} |x_m(i)| \\ &\geq \|x_m\|_1 - (c-1)(r/(c-1)) - r = \|x_m\|_1 - 2r \\ &\geq \|x_m\|_1 - \frac{2}{5}\|x_m\|_1 \text{ since } \|x_m\|_1 \geq 5r \\ &= \frac{3}{5}\|x_m\|_1.\end{aligned}$$

So (x_m) has a hump over $[c, d]$.

Proposition 6.16, Part 3

Proposition 6.16. If a sequence (x_n) in ℓ^1 converges weakly to x, then (x_n) converges to x with respect to the ℓ^1 norm. We take $\mathbb{F} = \mathbb{C}$.

Proof (Part 3). By starting with $k = 1$ we can produce a subsequence (x_{n_1}) which has a hump. Since $c \in \mathbb{N}$ above is arbitrary "we can push the humps out as far as we like" (page 138) to ensure that for $i \neq j$, the humps for x_{n_i} and x_{n_j} are disjoint intervals. Next for every element of an interval corresponding to a hump, $x_{n_k}(\ell) = re^{i\theta}$, define $y(\ell) = e^{-i\theta}$. For ℓ not in an interval corresponding to a hump, define $y(\ell) = 0$. Define $y = (y(1), y(2), \ldots)$. Then $\|y\|_\infty = 1$ and so $y \in \ell^\infty$ (the dual space of ℓ^1 by Theorem 6.1). For this $y \in \ell^\infty$, we consider the bounded linear functional φ_y on ℓ^1 where $\varphi_y(x_n) = \sum_{\ell=1}^{\infty} y(\ell) x_n(\ell)$.

Proposition 6.16, Part 4

Proof (Part 4). For any of the subsequences of (x_n) produced above (each with a hump), say (x_{n_k}), we have

$$\begin{aligned}|\varphi_y(x_{n_k})| &= \left|\sum_{\ell=1}^{\infty} y(\ell) x_{n_k}(\ell)\right| \\ &= \left|\sum_{\ell \in [c_k, d_k]} y(\ell) x_{n_k}(\ell) + \sum_{\ell \notin [c_k, d_k]} y(\ell) x_{n_k}(\ell)\right|\end{aligned}$$

where $[c_k, d_k]$ is the interval corresponding to the hump of x_{n_k} (notice that only a finite number of terms have been rearranged, so absolute convergence is not an issue) and then the quantity above is (by the Triangle Inequality)

$$\geq \left|\sum_{\ell \in [c_k, d_k]} y(\ell) x_{n_k}(\ell)\right| - \left|\sum_{\ell \notin [c_k, d_k]} y(\ell) x_{n_k}(\ell)\right| \qquad (*)$$

Proposition 6.16, Part 5

Proof (Part 5). Now in the hump, $y(\ell)x_{n_k}(\ell) = |x_{n_k}(\ell)|$ by the choice of $y(\ell)$. Outside the hump, $|y(\ell)x_{n_k}(\ell)| = |x_{n_k}(\ell)|$ and so

$$\left|\sum_{\ell\notin[c_k,d_k]} y(\ell)x_{n_k}(\ell)\right| \le \sum_{\ell\notin[c_k,d_k]} |y(\ell)||x_{n_k}(\ell) = \sum_{\ell\notin[c_k,d_k]} |x_{n_k}(\ell)| \le \frac{2}{5}\|x_{n_k}\|_1$$

since at least $\frac{3}{5}\|x_{n_k}\|_1$ is "in the hump"). So, by $(*)$,

$$|\varphi_y(x_{n_k})| \ge \sum_{\ell\in[c_k,d_k]} |x_{n_k}(\ell)| - \frac{2}{5}\|x_{n_k}\|_1 \ge \frac{3}{5}\|x_{n_k}\|_1 - \frac{2}{5}\|x_{n_k}\|_1 = \frac{1}{5}\|x_{n_k}\|_1 \ge r$$

by the choice of r.

Proposition 6.16, Part 6

Proposition 6.16. If a sequence (x_n) in ℓ^1 converges weakly to x, then (x_n) converges to x with respect to the ℓ^1 norm. We take $\mathbb{F} = \mathbb{C}$.

Proof (Part 6). So for $\varphi_y \in \ell^\infty$, we do not have $\varphi_y(x_{n_k}) \to 0$. Hence $\varphi_y(x_n)$ does not converge to 0, CONTRADICTING the fact that (x_n) converges weakly to 0. So the assumption that (x_n) converges weakly to 0. So the assumption that (x_n) does not converge to 0 with respect to the ℓ^1 norm is false.
Therefore, weak convergence in ℓ^1 implies convergence in ℓ^1. □