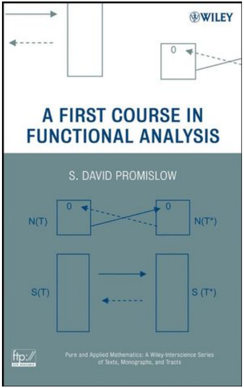
Introduction to Functional Analysis

Chapter 6. Duality

6.4. Weak and Weak* Convergence—Proofs of Theorems



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$$\|f(x) - f(x_n)\| = \|f(x - x_n)\| \leq \|f\| \|x - x_n\| < \|f\| \varepsilon / \|f\| = \varepsilon.$$

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For any sequence (x_n) which converges weakly to x, any sequence (y_n) which converges weakly to y, and any sequence of scalars (α_n) converging to α, we have:

(a) $(x_n + y_n)$ converges weakly to $x + y$,

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Proof. By the linearity of $f \in X^*$, without loss of generality we may assume $x = 0$. Suppose (x_n) converges weakly to $x = 0$.

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$$\sum_{i=c}^{d} |x_m(i)| = \|x_m\|_1 = \sum_{i=1}^{c-1} |x_m(i)| - \sum_{i=d+1}^{\infty} |x_m(i)|$$

$$\geq \|x_m\|_1 - (c-1)(r/(c-1)) - r = \|x_m\|_1 - 2r$$

$$\geq \|x_m\|_1 - \frac{2}{5}\|x_m\|_1 \text{ since } \|x_m\|_1 \geq 5r$$

$$= \frac{3}{5}\|x_m\|_1.$$

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Proposition 6.16, Part 4

Proof (Part 4). For any of the subsequences of (x_n) produced above (each with a hump), say (x_{n_k}), we have

$$\begin{aligned} |\varphi_y(x_{n_k})| &= \left|\sum_{\ell=1}^{\infty} y(\ell)x_{n_k}(\ell)\right| \\ &= \left|\sum_{\ell\in[c_k,d_k]} y(\ell)x_{n_k}(\ell) + \sum_{\ell\notin[c_k,d_k]} y(\ell)x_{n_k}(\ell)\right| \end{aligned}$$

where $[c_k, d_k]$ is the interval corresponding to the hump of x_{n_k} (notice that only a finite number of terms have been rearranged, so absolute convergence is not an issue) and then the quantity above is (by the Triangle Inequality)

$$\geq \left|\sum_{\ell\in[c_k,d_k]} y(\ell)x_{n_k}(\ell)\right| - \left|\sum_{\ell\notin[c_k,d_k]} y(\ell)x_{n_k}(\ell)\right| \qquad (*)$$

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Proof (Part 5). Now in the hump, $y(\ell)x_{n_k}(\ell) = |x_{n_k}(\ell)|$ by the choice of $y(\ell)$. Outside the hump, $|y(\ell)x_{n_k}(\ell)| = |x_{n_k}(\ell)|$ and so

$$\left|\sum_{\ell \notin [c_k, d_k]} y(\ell)x_{n_k}(\ell)\right| \leq \sum_{\ell \notin [c_k, d_k]} |y(\ell)||x_{n_k}(\ell) = \sum_{\ell \notin [c_k, d_k]} |x_{n_k}(\ell)| \leq \frac{2}{5}\|x_{n_k}\|_1$$

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Proof (Part 6). So for $\varphi_y \in \ell^\infty$, we do not have $\varphi_y(x_{n_k}) \to 0$. Hence $\varphi_y(x_n)$ does not converge to 0, CONTRADICTING the fact that (x_n) converges weakly to 0. So the assumption that (x_n) converges weakly to 0. So the assumption that (x_n) does not converge to 0 with respect to the ℓ^1 norm is false.

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Therefore, weak convergence in ℓ^1 implies convergence in ℓ^1. □

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