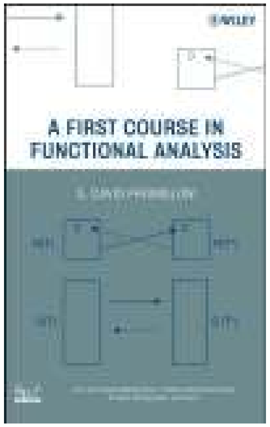
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.2. Banach Algebra—Proofs of Theorems



Lemma, Products and Inverses Lemma

Lemma. Products and Inverses Lemma.
Let x_1, x_2 be elements of a unital algebra.

(a) If x_1 has a left inverse and a right inverse, then x_1 is invertible.

(b) If x_1 and x_2 are invertible, then the product x_1x_2 is invertible.

(c) If the product x_1x_2 is invertible and $x_1x_2 = x_2x_1$, then each factor x_1 and x_2 is invertible.

Proof of (a). Suppose y is a right inverse of x_1 and z is a left inverse of x_1. Then $x_1y = zx_1 = e$ and so $z = ze = z(x_1y) = (zx_1)y = ey = y$. So $x_1^{-1} = z = y$. □

Lemma, Products and Inverses Lemma, Part (b)

Lemma. Products and Inverses Lemma.
Let x_1, x_2 be elements of a unital algebra.

(a) If x_1 has a left inverse and a right inverse, then x_1 is invertible.

(b) If x_1 and x_2 are invertible, then the product x_1x_2 is invertible.

(c) If the product x_1x_2 is invertible and $x_1x_2 = x_2x_1$, then each factor x_1 and x_2 is invertible.

Proof of (b). Since x_1 and x_2 are invertible, they have inverses, say x_1^{-1} and x_2^{-1} respectively. Then $(x_1x_2)(x_2^{-1}x_1^{-1}) = (x_2^{-1}x_1^{-1})(x_1x_2) = e$. So x_1x_2 is invertible and $(x_1x_2)^{-1} = x_2^{-1}x_1^{-1}$. □

Lemma, Products and Inverses Lemma, Part (c)

Lemma. Products and Inverses Lemma.
Let x_1, x_2 be elements of a unital algebra.

(a) If x_1 has a left inverse and a right inverse, then x_1 is invertible.

(b) If x_1 and x_2 are invertible, then the product x_1x_2 is invertible.

(c) If the product x_1x_2 is invertible and $x_1x_2 = x_2x_1$, then each factor x_1 and x_2 is invertible.

Proof of (c). Since x_1x_2 is invertible, let y be the inverse. Then $(x_1x_2)y = x_1(x_2y) = e$. So x_2y is a right inverse of x_1. Also $y(x_1x_2) = y(x_2x_1)$ since $x_1x_2 = x_2x_1 = (yx_2)x_1$ and so yx_2 is a left inverse of x. Hence by (a), x_1 is invertible and similarly x_2 is invertible. □