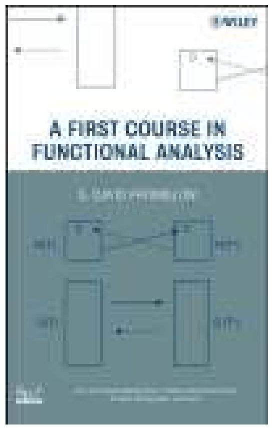
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.3. General Properties of the Spectrum—Proofs of Theorems



Theorem 8.5, The Spectral Mapping Theorem

Theorem 8.5. The Spectral Mapping Theorem.
Let p be a polynomial. Let X be a linear space. Then $\mu \in \sigma(p(x))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(x)$, where $x \in X$.

Proof. If p is the 0 polynomial, the claim is that $\mu \in \sigma(0) = \{0\}$ if and only if $\mu = p(\lambda) = 0$ for some $\lambda \in \sigma(x)$, so the result holds. So without loss of generality, p is not the 0 polynomial.
Suppose $\lambda \in \sigma(x)$. The polynomial with variable t of $p(t) = p(\lambda)$ has λ as a root, so it can be factored as $p(t) - p(\lambda) = (t - \lambda)q(t)$ where q is a polynomial. In the algebra this means
$p(x) - p(\lambda e) = (x - \lambda e)q(x) = q(x)(x - \lambda e)$ (the commutivity here will follow by writing out polynomial $q(x)$, distributing the x and λe which commute with the parts of $q(x)$ be definition of multiplication by x, scalar multiplication [which is commutative], and the fact that e is the unit and hence commutes).

Theorem 8.5, The Spectral Mapping Theorem (continued)

Proof (continued). Since the factors $(x - \lambda e)$ and $q(x)$ commute and $(x - \lambda e)$ is not invertible, then by the Products and Inverses Lemma of Section 8.2 (contrapositive of part (c)), the product $(x - \lambda e)q(x)$ is not invertible. So for $p(x) \in X$, we have that $p(x) - p(\lambda e) = p(x) - p(\lambda)e$ is not invertible, and so $p(\lambda) \in \sigma(p(x))$.
Now suppose $\mu \in \sigma(p(x))$. Let the zeros of $p(t) - \mu$ be $\mu_1, \mu_2, \ldots, \mu_n$ (Fundamental Theorem of Algebra over $\mathbb{C}$). Then in the algebra, $p(x) - \mu e = c(x - \mu_1 e)(x - \mu_2 e) \cdots (x - \mu_n e)$ for some nonzero scalar c. The left-hand side of this equation is not invertible since $\mu \in \sigma(p(x))$. The factors on the right-hand side all commute (as above). So by the Products and Inverses Lemma of Section 8.2 (the contrapositive of part (c), along with induction) at least one of the factors on the right-hand side is not invertible. That is, some $\mu_i \in \sigma(x)$ where $p(\mu_i)\mu$. Here μ_i is the λ of the statement of this result. □

Proposition 8.7

Proposition 8.7. Suppose x is invertible. Then $\lambda \in \sigma(x)$ if and only if $\lambda^{-1} \in \sigma(x^{-1})$.

Proof. Notice that x invertible implies $0 \notin \sigma(x)$. We have $x^{-1} - \lambda^{-1}e = -\lambda^{-1}(x - \lambda e)x^{-1}$ and $x - \lambda e$ is invertible for $\lambda \notin \sigma(x)$ since x^{-1} is invertible and $(x - \lambda e)$ and x^{-1} commute by the Products and Inverses Lemma . Similarly, if $\lambda \in \sigma(x)$ then $(x - \lambda e)$ is not invertible and $s^{-1} - \lambda^{-1}e$ is not invertible (by the contrapositive of the Products and Inverses Lemma part (c)) and so $\lambda^{-1} \in \sigma(x^{-1})$. □

Proposition 8.8

Proposition 8.8. Let X be a (complete) Banach algebra. If $\|e - x\| < 1$, then x is invertible.

Proof. Let x satisfy $\|e - x\| < 1$. Define

$$y = e + (e - x) + (e - x)^2 + (e - x)^3 + \cdots = \sum_{k=0}^{\infty}(e - x)^k$$

(we take $x^0 = e$ for all $x \in X$). Then this series converges absolutely (take norms of terms, and a geometric series with ration $\|e - x\| < 1$ results), and since the algebra is complete, the series converges. Define the partial sums $y_n = \sum_{k=0}^{n}(e - x)^k$. Then

$$y_n x = y_x - y_n + y_n x = y_n - y_n(e - x)$$

$$= \sum_{k=0}^{n}(e - x)^k - \sum_{k=0}^{n}(e - x)^{k+1} = e - (e - x)^{n+1}.$$

So $\lim_{n\to\infty}(y_n x) = \lim_{n\to\infty}(e - (e - x)^{n+1}) = e$ since $\|e - x\| < 1$. So $yx = e$ and similarly $xy = e$. That is, $x^{-1} = y$. □

Proposition 8.9

Proposition 8.9. The set of invertible elements of a Banach algebra is an open set.

Proof. Let x be invertible. Define $r = 1/\|x^{-1}\}$. Then for all y such that $\|x - y\| < r$ we have

$$\|e - x^{-1}y\| = \|x^{-1}(x - y)\| = \|x^{-1}\|\|x - y\| < \|x^{-1}\|r = 1$$

by the choice of r. So by Proposition 8.8, $x^{-1}y$ is invertible. So $y = x(x^{=1}y)$ is invertible and the result follows. □

Proposition 8.10

Theorem 8.10. Let X be a Banach algebra. Then for all $x \in X$, $\sigma(x)$ is a compact subset of $\mathbb{C}$.

Proof. Suppose $|\lambda| > \|x\|$. Then $\|e - (e - x/\lambda)\| = \|x\|/|\lambda| < 1$, so by Proposition 8.8, $e - x/\lambda$ is invertible, and hence so is $-\lambda(e - c/\lambda) = x - \lambda e$. So $x - \lambda e$ is NOT invertible only for $\lambda \le \|x\|$. That is, the spectrum of x is a subset of $\overline{B}(\|x\|)$: $\sigma(x) \subseteq \overline{B}(\|x\|)$. So $\sigma(x)$ is a bounded subset of $\mathbb{C}$.
Of $\lambda \notin \sigma(x)$ then $s - \lambda e$ is invertible and since the set of invertible elements is open by Proposition 8.9, there exists $\varepsilon > 0$ such that for all $\mu \in \mathbb{C}$ where $|\lambda - \mu| < \varepsilon$, we have $x - \mu e$ is invertible. That is, $B(\lambda; \varepsilon) \subseteq \mathbb{C} \setminus \sigma(x)$. So $\mathbb{C} \setminus \sigma(x)$ is open and $\sigma(x)$ is closed.
Therefore, $\sigma(x)$ is a closed and bounded set in $\mathbb{C}$ and hence $\sigma(x)$ is compact. □

Proposition 8.11

Proposition 8.11. Let X be a Banach algebra. Then for any $x \in X$, the spectral radius of x satisfies

$$r(x) \le \inf\{\|x^n\|^{1/n} \mid n \in \mathbb{N}\}.$$

Proof. As shown in the proof of Theorem 8.10, since $\sigma(x) \subseteq \overline{B}(\|x\|)$, then $r(x) \le \|x\|$. So for any $n \in \mathbb{N}$ (replacing arbitrary vector x with vector x^n), $r(x^n) \le \|x^n\|$.
Consider the polynomial $p(x) = x^n$. By the Spectral Mapping Theorem, $\mu \in \sigma(p(x)) - \sigma(x^n)$ if and only if $\mu = p(\lambda) = \lambda^n$ for some $\lambda \in \sigma(x)$. So the spectral radius of x and x^n are related as: $(r(x))^n = r(x^n)$. and for all n, $r(x) = f(x^n)^{1/n}$. Hence,

$$r(x) \le \inf\{(r(x^n))^{1/n} \mid n \in \mathbb{N}\}.$$

□

Theorem 8.12

Theorem 8.12. If (a_n) is a submultiplicative sequence of positive real numbers, then $(a_n^{1/n})$ converges to $\inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$.

Proof. Let $a = \inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$. For any $b > a$ we can choose $k \in \mathbb{N}$ so that $a_k^{1/k} < b$ (by the definition of a in terms of an infimum). For fixed $n \in \mathbb{N}$ where $n > k$, write $n = pk + r$ where $p \in \mathbb{N}$ and $0 \le r \le k - 1$. Since (a_n) is submultiplicative, we have inductively:

$$a_n = a_{pk+r} = a_{k+k+\cdots+k+r} \le a_k a_k \cdots a_k a_r = (a_k)^p a_r.$$

Now $n = pk + r$ implies that $n - r = pk$ and hence $p/n = (n-r)/(nk) = (1/k)(1 - r/n)$. So

$$a_n^{1/n} \le a_k^{p/n} a_r^{1/n} = (a_k^{1/k})^{1-r/n} a_r^{1/n} < b^{1-r//n} c^{1/n} \qquad (*)$$

(by the choice of b) where $c = \max\{a_r \mid r = 1, 2, \ldots, k-1\}$. Since each a_n is nonnegative, $c^{1/n} \to 1$ as $n \to \infty$. Since $r \le k - 1$ (i.e., r is bounded) then $1 - r/n \to 1$ as $n \to \infty$. So $b^{1-r/n} c^{1/n} \to b$ as $n \to \infty$.

Theorem 8.12 (continued)

Theorem 8.12. If (a_n) is a submultiplicative sequence of positive real numbers, then $(a_n^{1/n})$ converges to $\inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$.

Proof (continued). Since a is the infimum of all $a_n^{1/n}$, we have $a \le a_n^{1/n}$ for all n. Let $\varepsilon > 0$. The above argument starts with an arbitrary $b > a$, so choose b such that $b - a < \varepsilon/2$. Also, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $|b^{1-r/n} c^{1/n} - b| < \varepsilon/2$. Next, $(*)$ holds for any $n > k$, so for all $n \ge \max\{N, k\}$, we have $a \le a_n^{1/n} < b + \varepsilon/2$, and so for all $n, m \ge \max\{N, k\}$ we have $|a_n^{1/n} - a_m^{1/m}| < b + \varepsilon/2 - a < \varepsilon$. So $(a_n^{1/n})$ is a Cauchy sequence of real numbers. So $(a_n^{1/n}) \to a'$ for some $a' \le b$. Since $b > a$ was chosen arbitrarily, we have $a' \le a = \inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$, but since a' is the limit of $(a_n^{1/n})$, then $a' = a$. That is,

$$a = \inf\{a_n^{1/n} \mid n \in \mathbb{N}\} = a' = \lim_{n\to\infty} (a_n^{1/n}).$$

□