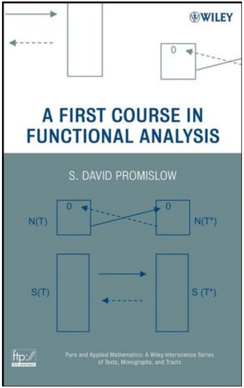
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.3. General Properties of the Spectrum—Proofs of Theorems



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Proof. If p is the 0 polynomial, the claim is that $\mu \in \sigma(0) = \{0\}$ if and only if $\mu = p(\lambda) = 0$ for some $\lambda \in \sigma(x)$, so the result holds. So without loss of generality, p is not the 0 polynomial.

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Theorem 8.5, The Spectral Mapping Theorem (continued)

Proof (continued). Since the factors $(x - \lambda e)$ and $q(x)$ commute and $(x - \lambda e)$ is not invertible, then by the Products and Inverses Lemma of Section 8.2 (contrapositive of part (c)), the product $(x - \lambda e)q(x)$ is not invertible. So for $p(x) \in X$, we have that $p(x) - p(\lambda e) = p(x) - p(\lambda)e$ is not invertible, and so $p(\lambda) \in \sigma(p(x))$.

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$$y = e + (e - x) + (e - x)^2 + (e - x)^3 + \cdots = \sum_{k=0}^{\infty} (e - x)^k$$

(we take $x^0 = e$ for all $x \in X$). Then this series converges absolutely (take norms of terms, and a geometric series with ration $\|e - x\| < 1$ results), and since the algebra is complete, the series converges.

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$$y_n x = y_x - y_n + y_n x = y_n - y_n(e - x)$$

$$= \sum_{k=0}^{n}(e - x)^k - \sum_{k=0}^{n}(e - x)^{k+1} = e - (e - x)^{n+1}.$$

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So $\lim_{n\to\infty}(y_n x) = \lim_{n\to\infty}(e - (e - x)^{n+1}) = e$ since $\|e - x\| < 1$. So $yx = e$ and similarly $xy = e$. That is, $x^{-1} = y$. $\square$

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Proposition 8.9. The set of invertible elements of a Banach algebra is an open set.

Proof. Let x be invertible. Define $r = 1/\|x^{-1}\}$. Then for all y such that $\|x - y\| < r$ we have

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Proposition 8.11. Let X be a Banach algebra. Then for any $x \in X$, the spectral radius of x satisfies

$$r(x) \leq \inf\{\|x^n\|^{1/n} \mid n \in \mathbb{N}\}.$$

Proof. As shown in the proof of Theorem 8.10, since $\sigma(x) \subseteq \overline{B}(\|x\|)$, then $r(x) \leq \|x\|$.

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$$a_n = a_{pk+r} = a_{k+k+\cdots+k+r} \leq a_k a_k \cdots a_k a_r = (a_k)^p a_r.$$

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Proof. Let $a = \inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$. For any $b > a$ we can choose $k \in \mathbb{N}$ so that $a_k^{1/k} < b$ (by the definition of a in terms of an infimum). For fixed $n \in \mathbb{N}$ where $n > k$, write $n = pk + r$ where $p \in \mathbb{N}$ and $0 \le r \le k - 1$. Since (a_n) is submultiplicative, we have inductively:

$$a_n = a_{pk+r} = a_{k+k+\cdots+k+r} \le a_k a_k \cdots a_k a_r = (a_k)^p a_r.$$

Now $n = pk + r$ implies that $n - r = pk$ and hence $p/n = (n-r)/(nk) = (1/k)(1 - r/n)$. So

$$a_n^{1/n} \le a_k^{p/n} a_r^{1/n} = (a_k^{1/k})^{1-r/n} a_r^{1/n} < b^{1-r//n} c^{1/n} \qquad (*)$$

(by the choice of b) where $c = \max\{a_r \mid r = 1, 2, \ldots, k-1\}$.

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(by the choice of b) where $c = \max\{a_r \mid r = 1, 2, \ldots, k - 1\}$. Since each a_n is nonnegative, $c^{1/n} \to 1$ as $n \to \infty$. Since $r \leq k - 1$ (i.e., r is bounded) then $1 - r/n \to 1$ as $n \to \infty$. So $b^{1-r/n} c^{1/n} \to b$ as $n \to \infty$.

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Proof (continued). Since a is the infimum of all $a_n^{1/n}$, we have $a \leq a_n^{1/n}$ for all n. Let $\varepsilon > 0$. The above argument starts with an arbitrary $b > a$, so choose b such that $b - a < \varepsilon/2$. Also, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|b^{1-r/n}c^{1/n} - b| < \varepsilon/2$.

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$$a = \inf\{a_n^{1/n} \mid n \in \mathbb{N}\} = a' = \lim_{n\to\infty}(a_n^{1/n}).$$

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