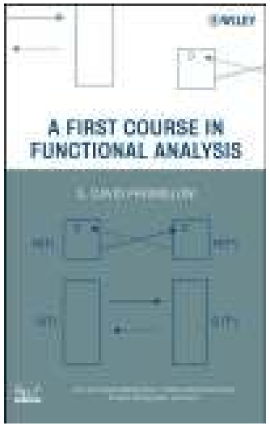
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.4. Numerical Range—Proofs of Theorems



Proposition 8.17

Proposition 8.17. For H a Hilbert space and $T \in \mathcal{B}(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of T.

Proof. If $\lambda \in \sigma(T)$. If λ is an eigenvalue, choose a unit vector x so that $Tx = \lambda x$. Then

$$\langle Tx, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle - \lambda\|x\|^2 = \lambda(1) = \lambda.$$

So λ is in the numerical range of T for all eigenvalues of T. This covers the point spectrum of T (see Section 8.1).

If $T - \lambda I$ is not bounded below (recall from Section 3.4 that T is bounded below if there is a $k > 0$ such that for all unit vectors x we have $\|Tx\| \geq k$) then there is a sequence (x_n) of unit vectors such that $(T - \lambda I)x_n \to 0$. Also,

$$\langle (T - \lambda I)x_n, x_n\rangle = \langle Tx_n, x_n\rangle - \langle x_n, x_n\rangle = \langle Tx_n, x_n\rangle - \lambda.$$

So $\lim(\langle Tx_{n,n}\rangle - \lambda) = 0$ and $\lambda = \lim\langle Tx_n, x_n\rangle$.

Proposition 8.17 (continued)

Proposition 8.17. For H a Hilbert space and $T \in \mathcal{B}(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of T.

Proof (continued). So λ is in the closure of the numerical range in the case that $T - \lambda I$ is not bounded below. This potentially includes part of the continuous spectrum and part of the residual spectrum

If $T\lambda I$ is bounded below but the closure of the range $R(T - \lambda U) \neq H$, then by Proposition 4.27(a) (which says for T, $N(T^*) = R(T)^\perp$ where $N(T^*)$ is the null space of T) there is $y \in N((T - \lambda I)^*) = N(T^* - \overline{\lambda} I)$ (by Theorem 4.26(b)). Then $(T^* - \overline{\lambda} I)(y) = T^*y - \overline{\lambda}y = 0$ and

$$0 = \langle 0, y\rangle = \langle (T^* - \overline{\lambda} I)y, y\rangle = \langle T^*y - \overline{\lambda}y, y\rangle = \langle T^*y, y\rangle - \overline{\lambda}\langle y, y\rangle$$
$$= \langle T^*y, y\rangle - \overline{\lambda}(1) = \langle T^*y, y\rangle - \overline{\lambda}$$

or $\overline{\lambda} - \langle T^*y, y\rangle = \langle y, Ty\rangle = \overline{\langle Ty, y\rangle}$ and hence $\lambda = \langle Ty, y\rangle$ is in the numerical range. This covers the rest of the continuous and residual spectrums. □