Chapter 8. The Spectrum
8.4. Numerical Range—Proofs of Theorems
Proposition 8.17
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**Proposition 8.17.** For $H$ a Hilbert space and $T \in \mathcal{B}(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of $T$.

**Proof.** If $\lambda \in \sigma(T)$. If $\lambda$ is an eigenvalue, choose a unit vector $x$ so that $Tx = \lambda x$. 

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$$\langle Tx, x \rangle = \langle \lambda x, x \rangle = \lambda \langle x, x \rangle - \lambda \|x\|^2 = \lambda(1) = \lambda.$$

So $\lambda$ is in the numerical range of $T$ for all eigenvalues of $T$. This covers the point spectrum of $T$ (see Section 8.1).
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If $T - \lambda I$ is not bounded below (recall from Section 3.4 that $T$ is bounded below if there is a $k > 0$ such that for all unit vectors $x$ we have $\|Tx\| \geq k$) then there is a sequence $(x_n)$ of unit vectors such that $(T - \lambda I)x_n \to 0$. 
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$$\langle (T - \lambda I)x_n, x_n \rangle = \langle Tx_n, x_n \rangle - \langle x_n, x_n \rangle = \langle Tx_n, x_n \rangle - \lambda.$$

So $\lim(\langle Tx_n, x_n \rangle - \lambda) = 0$ and $\lambda = \lim \langle Tx_n, x_n \rangle$. 

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Proposition 8.17 (continued)

Proposition 8.17. For $H$ a Hilbert space and $T \in B(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of $T$.

Proof (continued). So $\lambda$ is in the closure of the numerical range in the case that $T - \lambda I$ is not bounded below. This potentially includes part of the continuous spectrum and part of the residual spectrum.

If $T \lambda I$ is bounded below but the closure of the range $R(T - \lambda U) \neq H$, then by Proposition 4.27(a) (which says for $T$, $N(T^*) = R(T)^\perp$ where $N(T^*)$ is the null space of $T$) there is $y \in N((T - \lambda I)^*) = N(T^* - \lambda I)$ (by Theorem 4.26(b)).
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$$0 = \langle 0, y \rangle = \langle (T^* - \overline{\lambda}I)y, y \rangle = \langle T^*y - \overline{\lambda}y, y \rangle = \langle T^*y, y \rangle - \overline{\lambda}\langle y, y \rangle$$

$$= \langle T^*y, y \rangle - \overline{\lambda}(1) = \langle T^*y, y \rangle - \overline{\lambda}$$

or $\overline{\lambda} - \langle T^*y, y \rangle = \langle y, Ty \rangle = \overline{\langle Ty, y \rangle}$ and hence $\lambda = \langle Ty, y \rangle$ is in the numerical range. This covers the rest of the continuous and residual spectrums.
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