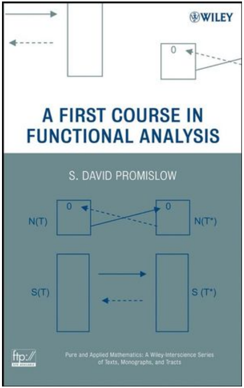
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.4. Numerical Range—Proofs of Theorems



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If $T - \lambda I$ is not bounded below (recall from Section 3.4 that T is bounded below if there is a $k > 0$ such that for all unit vectors x we have $\|Tx\| \geq k$) then there is a sequence (x_n) of unit vectors such that $(T - \lambda I)x_n \to 0$.

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$$\langle (T - \lambda I)x_n, x_n\rangle = \langle Tx_n, x_n\rangle - \langle x_n, x_n\rangle = \langle Tx_n, x_n\rangle - \lambda.$$

So $\lim(\langle Tx_{n,n}\rangle - \lambda) = 0$ and $\lambda = \lim\langle Tx_n, x_n\rangle$.

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Proof (continued). So λ is in the closure of the numerical range in the case that $T - \lambda I$ is not bounded below. This potentially includes part of the continuous spectrum and part of the residual spectrum

If $T \lambda I$ is bounded below but the closure of the range $R(T - \lambda U) \neq H$, then by Proposition 4.27(a) (which says for T, $N(T^*) = R(T)^\perp$ where $N(T^*)$ is the null space of T) there is $y \in N((T - \lambda I)^*) = N(T^* - \overline{\lambda} I)$ (by Theorem 4.26(b)).

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$$0 = \langle 0, y \rangle = \langle (T^* - \overline{\lambda} I) y, y \rangle = \langle T^* y - \overline{\lambda} y, y \rangle = \langle T^* y, y \rangle - \overline{\lambda} \langle y, y \rangle$$
$$= \langle T^* y, y \rangle - \overline{\lambda}(1) = \langle T^* y, y \rangle - \overline{\lambda}$$

or $\overline{\lambda} - \langle T^* y, y \rangle = \langle y, Ty \rangle = \overline{\langle Ty, y \rangle}$ and hence $\lambda = \langle Ty, y \rangle$ is in the numerical range. This covers the rest of the continuous and residual spectrums. □

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