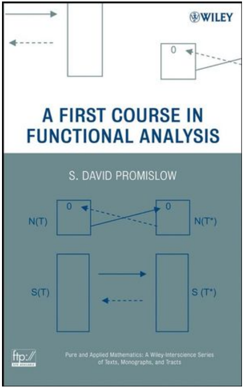
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.5. Spectrum of a Normal Operator—Proofs of Theorems



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(a) If T is self-adjoint then $\sigma(T) \subseteq \mathbb{R}$.

(b) If T is a positive operator then $\sigma(T)$ consists of nonnegative real numbers.

(c) If T is a projection then $\sigma(T) \subseteq \{0, 1\}$.

(d) If T is a unitary operator, then $\sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. (a) By Proposition 8.17, the spectrum of T, $\sigma(T)$ is contained in the closure of $\{\langle Tx, x\rangle \mid \|x\| = 1\}$. Since T is self-adjoint then $T = T^*$ and $\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ and so $\langle Tx, x\rangle$ is real for all $\|x\| = 1$ (for all x, in fact). So $\sigma(T)$ consists only of real numbers.

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(b) For positive operator, $\langle Tx, x\rangle \geq 0$ for all x. By Proposition 8.17, $\sigma(T)$ is contained in the closure of $\{\langle Tx, x\rangle \mid \|x\| = 1\}$ and so all $\sigma(T)$ consists only of nonnegative (real) numbers.

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Proof (continued). (d) Let U be a unitary operator. By Proposition 4.34, U is an isometry so that $\|Ux\| = \|x\|$. So by the Cauchy-Schwarz Inequality, $|\langle Ux, x\rangle| \leq \|Ux\|\|x\| = \|x\|^2 = 1$. Since by Proposition 8.7, $\lambda \in \sigma(U)$ if and only if $\lambda^{-1} \in \sigma(U^{-1})$. With U unitary, $U^* = U^{-1}$ is also unitary. So $\sigma(U^{-1}) \subset \{z \in \mathbb{C} \mid |z| \leq 1\}$.

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Lemma 8.19

Lemma 8.19. If T is a self-adjoint operator on a Hilbert space H, then for all unit vectors x and y in H, we have $\mathrm{Re}(\langle Tx, y\rangle) \leq w(T)$.

Proof. Let T be self-adjoint and let x and y be unit vectors. Then

$$\begin{aligned}
\langle T(x+y), x+y\rangle &= \langle Tx+Ty, x+y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \overline{\langle T^*x, y\rangle} + \langle Ty, y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \overline{\langle Tx, y\rangle} + \langle Ty, y\rangle \text{ since } T^* = T \\
&= \langle Tx, x\rangle + 2\mathrm{Re}(\langle Tx, y\rangle) + \langle Ty, \rangle.
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Similarly,

$$\langle T(x-y), x-y\rangle = \langle Tx, x\rangle - 2\mathrm{Re}(\langle Tx, y\rangle) + \langle Ty, y\rangle.$$

Taking the difference of these results gives

$$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 4\mathrm{Re}(\langle Tx, y\rangle)$$

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Lemma 8.19 (continued)

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$$
\begin{aligned}
\mathrm{Re}(\langle Tx, y\rangle) &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) \\
&\le \frac{1}{4}(\|T(x+y)\|\|x+y\|_{\|}T(x-y)\|\|x-y\|) \\
&\qquad \text{by the Cauchy-Schwarz Inequality} \\
&\le \frac{1}{4}(\|T\|\|x+y\|^2 + \|T\|\|x-y\|^2) \\
&\le \frac{1}{4}w(t)(\|x+y\|^2 + \|x-y\|^2) \text{ since } w(t) \le \|T\| \\
&= \frac{1}{4}w(t)(2(\|x\|^2 + \|y\|^2)) \\
&\qquad \text{by the Parallelogram Law (Proposition 4.5)} \\
&= w(t) \text{ since } \|x\| = \|y\| = 1.
\end{aligned}
$$

□

Proposition 8.20

Proposition 8.20. If T is a self-adjoint operator on a Hilbert space, then $w(T) = \|T\|$.

Proof. We have $w(T) \leq \|T\|$ in general (see the note before Proposition 8.17). Let x be any unit vector and define $y = Tx/\|Tx\|$ if $tx \neq 0$. Then

$$\|Tx\| = \|Tx\|/\|Tx\|^2 = \langle Tx, y\rangle \leq w(T) \text{ by Lemma 8.19.}$$

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Also, if $Tx = 0$ then trivially $\|Tx\| = 0 \leq w(T)$. So

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Proposition 8.21. For a self-adjoint operator T on a Hilbert space, either $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$.

Proof. Since T is self-adjoint, by Proposition 8.20 $w(T) = \|T\|$ (and so $w(T)$ is finite since T is bounded), say $\|T\| = \lambda$. By Proposition 8.18(a), the numerical range of T is real, so λ is either the supremum of the numerical range of $-\lambda$ is the infimum of the numerical range.

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First, suppose λ is the supremum. For any unit vector x we have

$$\begin{aligned} \|(T-\lambda I)x\|^2 &= \|Tx\|^2 - 2\lambda\langle Tx, x\langle + \lambda^2 \\ &= \langle Tx, Tx\rangle - 2\lambda\langle Tx, x\rangle + \lambda^2\langle x, \rangle \\ &= \langle Tx, Tx\rangle - 2\lambda\langle Tx, x\rangle + \lambda^2 \text{ since } \|x\| = 1 \end{aligned}$$

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Proof (continued). So $T - \lambda I$ is not bounded below (see Section 3.4). If T is not injective, then λ is in the point spectrum of T. If $T - \lambda I$ is injective and $T - \lambda I$ is not bounded below, then λ is in the continuous spectrum or residual spectrum of T (see Section 8.1). Either way, $T \in \sigma(T)$, as claimed.

Second, suppose $-\lambda$ is the infimum of the numerical range of T. We have $\sigma(T) = -\sigma(-T)$. Then λ is the supremum of the numerical range of $-T$. By the result above, $\lambda \in \sigma(-T)$ and so $-\lambda \in -\sigma(-T) = \sigma(T)$, as claimed. □

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Proof. By Proposition 2.8, $\|ST\| \leq \|S\|\|T\|$ so, by induction, $\|T^n\| \leq \|T\|^n$.

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$$\begin{aligned} \|T\|^{n+k} &= \|T^{n+k}\| \leq \|T^n\|\|T^k\| \text{ by Proposition 2.8} \\ &\leq \|T^n\|\|T\|^k \text{ as shown above.} \end{aligned}$$

So $\|T\|^n \leq \|T^n\|$ for $n \in \mathbb{N}$.

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Proof (continued). Let $m \in \mathbb{N}$ and suppose $\|S^m\| = \|S\|^m$ for all normal S. Then

$$\begin{aligned} \|T^{2m}\| &= \|(T^*)^m T^m\| \text{ since } T \text{ is self adjoint} \\ &= \|(T^m)^* T^m\| \text{ since } T \text{ is self adjoint} \\ &= \|T^m\|^2 \text{ by Theorem 4.2.6(f)} \\ &= \|T\|^{2m} \text{ by the hypothesis that } \|S^m\| = \|S\|^m. \end{aligned}$$

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Since $\|T^m\| = \|T\|^m$ trivially when $m = 1$, then inductively this holds for all even m and, by the previous paragraph, holds for all $m \in \mathbb{N}$. So the claim holds for T self adjoint.

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$$\begin{aligned}
\|T^*T\|^n &= \|(T^*T)^n\| \text{ by the previous paragraph} \\
&= \|(T^*)^n T^n\| \text{ since } T \text{ is normal} \\
&\leq \|(T^*)^n\| \|T^n\| \text{ by Proposition 2.8} \\
&\leq \|T^*\|^n \|T^n\| \text{ by the first paragraph} \\
&= \|T\|^n \|T^n\| \text{ by Theorem 4.26(d).}
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Proof (continued). Now suppose T is normal (so by definition, $T^*T = TT^*$). Now $(T^*T)^* = T^*T^{**} = T^*T$ by Theorem 4.26(c) and (e), so T^*T is self adjoint. So

$$\begin{aligned} \|T^*T\|^n &= \|(T^*T)^n\| \text{ by the previous paragraph} \\ &= \|(T^*)^nT^n\| \text{ since } T \text{ is normal} \\ &\leq \|(T^*)^n\|\|T^n\| \text{ by Proposition 2.8} \\ &\leq \|T^*\|^n\|T^n\| \text{ by the first paragraph} \\ &= \|T\|^n\|T^n\| \text{ by Theorem 4.26(d).} \end{aligned}$$

Next, $\|T^*T\|^n = (\|T\|^2)^n = \|T\|^{2n}$ by Theorem 4.26(f). So $\|T\|^{2n} \leq \|T\|^n\|T^n\|$ and $\|T\|^n \leq \|T^n\|$. Combining this with the first paragraph, we have $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$ and the claim holds for T normal. □

Proposition 8.22 (continued 2)

Proposition 8.22. If T is a self adjoint or normal operator on a Hilbert space, then $\|T^n\| = \|T\|^n$.

Proof (continued). Now suppose T is normal (so by definition, $T^*T = TT^*$). Now $(T^*T)^* = T^*T^{**} = T^*T$ by Theorem 4.26(c) and (e), so T^*T is self adjoint. So

$$\begin{aligned} \|T^*T\|^n &= \|(T^*T)^n\| \text{ by the previous paragraph} \\ &= \|(T^*)^n T^n\| \text{ since } T \text{ is normal} \\ &\leq \|(T^*)^n\| \|T^n\| \text{ by Proposition 2.8} \\ &\leq \|T^*\|^n \|T^n\| \text{ by the first paragraph} \\ &= \|T\|^n \|T^n\| \text{ by Theorem 4.26(d).} \end{aligned}$$

Next, $\|T^*T\|^n = (\|T\|^2)^n = \|T\|^{2n}$ by Theorem 4.26(f). So $\|T\|^{2n} \leq \|T\|^n \|T^n\|$ and $\|T\|^n \leq \|T^n\|$. Combining this with the first paragraph, we have $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$ and the claim holds for T normal. □

Theorem 8.23

Theorem 8.23. If T is a normal operator on a Hilbert space, then $r(T) = \|T\|$.

Proof. When X is a normed linear space, then $\mathcal{B}(X)$ is a Banach algebra (see the last example in the class notes for Section 8.2). So for H a Hilbert space, $\mathcal{B}(H)$ is a Banach algebra. So for $T \in \mathcal{B}(H)$, by Theorem 8.15, the spectral radius is $r(T) = \inf \|T^n\|^{1/n}$. So by Proposition 8.22,

$$r(T) = \inf \|T^n\|^{1/n} = \inf(\|T\|^n)^{1/n} = \inf \|T\| = \|T\|.$$

□

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$$r(T) = \inf \|T^n\|^{1/n} = \inf(\|T\|^n)^{1/n} = \inf \|T\| = \|T\|.$$

$\square$

Proposition 8.24

Proposition 8.24. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be self-adjoint. Then eigenspaces corresponding to distinct eigenvalues of T are orthogonal.

Proof. Consider $Tx = \lambda x$ and $Ty = \mu y$ for distinct eigenvalues λ and μ with respective eigenvectors x and y. Since T is self adjoint by hypothesis, then by Proposition 8.18(a), λ and μ are real.

Proposition 8.24

Proposition 8.24. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be self-adjoint. Then eigenspaces corresponding to distinct eigenvalues of T are orthogonal.

Proof. Consider $Tx = \lambda x$ and $Ty = \mu y$ for distinct eigenvalues λ and μ with respective eigenvectors x and y. Since T is self adjoint by hypothesis, then by Proposition 8.18(a), λ and μ are real. Then

$$
\begin{aligned}
\lambda\langle x, y\rangle &= \langle x, y\rangle = \langle Tx, y\rangle \\
&= \langle x, T^*y\rangle = \langle x, Ty\rangle \text{ since } T \text{ is self adjoint} \\
&= \langle x, \mu y\rangle - \mu\langle x, y\rangle \text{ since } \mu \text{ is real.}
\end{aligned}
$$

Since $\lambda \neq \mu$, we have $\langle x, y\rangle = 0$. So if x and y are elements of eigenspaces associated with distinct eigenvalues, then x and y are orthogonal. $\square$

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Proposition 8.25

Proposition 8.25. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be self-adjoint. If H is separable, then the number of distinct eigenvalues of T is either finite or countably infinite.

Proof. For distinct eigenvalues $\{\lambda_i \mid i \in I\}$, choose a corresponding set of unit eigenvectors $\{x_i \mid i \in I\}$. By Proposition 8.24, $\{x_i \mid i \in I\}$ is an orthonormal set. By Theorem 4.21, since H is separable by hypothesis, $\{x_i \mid i \in I\}$ is either finite or countably infinite. Hence, $\{\lambda_i \mid i \in I\}$ is either finite or countably infinite. □

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