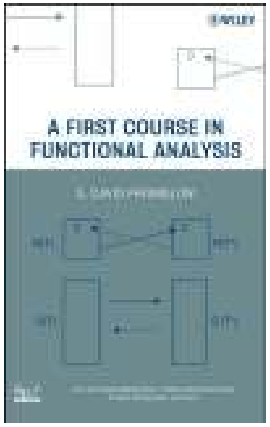
Introduction to Functional Analysis

Chapter 8. The Spectrum

8.6. Functions of Operators—Proofs of Theorems



Proposition 8.26

Proposition 8.26. Let T be a bounded, self adjoint operator on a Hilbert space H. Then the mapping that sends polynomials $p \in C(\sigma(T))$ (continuous functions on the spectrum of T) with the sup norm into $p(T) \in \mathcal{B}(H)$, sending $f \in C(\sigma(T))$ to $f(T)$ such that the following properties hold. For all $f, g \in C(\sigma(T))$ and any $\alpha \in \mathbb{C}$,

(i) $(f+g)(T) = f(T) + g(T)$ and $(\alpha f)(T) = \alpha f(T)$,
(ii) $\overline{f}(T) = (f(T))^*$ where $\overline{f}$ is the conjugate of function f,
(iii) $(fg)(T) = f(T)g(T)$, and
(iv) for any $S \in \mathcal{B}(H)$ such that $ST = TS$ we have $Sf(T) = f(T)S$.

Proof. Since $C(\sigma(T))$ has the sup norm then the norm of $p \in C(\sigma(T))$ is $\|p\| = \sup\{|p(\lambda)| \mid \lambda \in \sigma(T)\}$. By the Spectral Mapping Theorem (Theorem 8.5), $\mu = p(\lambda)$ for some $\lambda \in \sigma(T)$ if and only if $\mu \in \sigma(p(T))$, so we also have $\|p\| = \sup\{|\mu| \mid \mu \in \sigma(p(T))\} = r(p(T))$ (where $r(p(T))$ is the spectral radius of $p(r)$; the equality holds by the definition of spectral radius).

Proposition 8.26 (continued 1)

Proof (continued). Since T is a normal operator, by Theorem 8.23, $\|p\| = r(p(T)) = \|p(T)\|$. Therefore the mapping of p to $p(T)$ is an isometry.

The set of polynomials in $C(\sigma(T))$ form a dense subset of $C(\sigma(T))$ by the Stone-Weierstrass Theorem (see Appendix B of Promislow or Section 12.3 of Royden and Fitzpatrick). By Theorem 2.20, the mapping of the set of polynomials in $C(r(T))$ to $\mathcal{B}(H)$ (notice $\mathcal{B}(H)$ is a Banach space by Theorem 2.15) can be extended to an isometry defined on all of $X(\sigma(T))$. We now prove each of the four claims.

Let $f, g \in C(\sigma(T))$. Then, since the set of polynomials is dense in $C(\sigma(T))$ then there are sequence of polynomials (p_n) and (q_n) with $(p_n) \to f$ and $(q_n) \to g$ (the convergence is uniform since $C(\sigma(T))$ is equipped with the sup norm).

Proposition 8.26 (continued 2)

Proof (continued). **(i)** Now $p_n + q_n = f + g$ and so

$$(f+g)(T) = \left(\lim_{n\to\infty}(p_n+q_n)(T)\right) = \lim_{n\to\infty}(p_n(T)+q_n(T))$$
$$= \lim_{n\to\infty} p_n(T) + \lim_{n\to\infty} q_n(T) = f(T) + g(T).$$

For $\alpha \in \mathbb{C}$,

$$(\alpha f)(T) = \lim_{n\to\infty}(\alpha p_n)(T) = \lim_{n\to\infty} \alpha p_n(T) = \alpha \lim_{n\to\infty} p_n(T) = \alpha f(T).$$

(ii) We have

$$\begin{aligned} \overline{f}(T) &= \left(\overline{\lim_{n\to\infty} p_n}\right)(T) \lim_{n\to\infty}(\overline{p}_n(T)) \\ &= \lim_{n\to\infty}(\overline{p}_n(T^*)) \text{ since } T = T^* \\ &= \lim_{n\to\infty}(p_n(T))^* \text{ by Theorem 4.26 (consider } p_n \text{ in terms of its} \\ &\quad \text{coefficients and the fact that } T = T^* \\ &= \left(\lim_{n\to\infty} p_n(T)\right)^* = (f(T))^*. \end{aligned}$$

Proposition 8.26 (continued 3)

Proof (continued). (iii) We have

$$\begin{aligned}(fg)(T) &= \lim_{n\to\infty}(p_n q_n)(T) = \lim_{n\to\infty}(p_n(T)q_n(T)) \\ &= \lim_{n\to\infty} p_n(T) \lim_{n\to\infty} q_n(T) = f(T)g(T).\end{aligned}$$

(iv) Let $X \in \mathcal{B}(H)$ such that $ST = TS$. Notice that

$S(a_i T^i + a_j T^j) = a_i S(T^i) + a_j S(T^j) = a_i T^i S + a_j T^j S = (a_i T^i + a_j T^j)S.$

So for a polynomial p, $Sp(T) = p(T)S$. Hence

$$\begin{aligned}Sf(T) &= S\left(\lim_{n\to\infty} p_n(T)\right) \\ &= \lim_{n\to\infty} Sp_n(T) \text{ since } S \text{ is continuous} \\ &= \lim_{n\to\infty}(p_n(T)S) \text{ as argued above} \\ &= \left(\lim_{n\to\infty} p_n(T)\right) S \text{ since } S \text{ is continuous} \\ &= f(T)S.\end{aligned}$$

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