Chapter 8. The Spectrum

8.6. Functions of Operators—Proofs of Theorems
1 Proposition 8.26
Proposition 8.26. Let $T$ be a bounded, self adjoint operator on a Hilbert space $H$. Then the mapping that sends polynomials $p \in C(\sigma(T))$ (continuous functions on the spectrum of $T$) with the sup norm into $p(T) \in \mathcal{B}(H)$, sending $f \in C(\sigma(T))$ to $f(T)$ such that the following properties hold. For all $f, g \in C(\sigma(T))$ and any $\alpha \in \mathbb{C}$,

(i) $(f + g)(T) = f(T) + g(T)$ and $(\alpha f)(T) = \alpha f(T)$,
(ii) $\overline{f}(T) = (f(T))^*$ where $\overline{f}$ is the conjugate of function $f$,
(iii) $(fg)(T) = f(T)g(T)$, and
(iv) for any $S \in \mathcal{B}(H)$ such that $ST = TS$ we have $Sf(T) = f(T)S$.

Proof. Since $C(\sigma(T))$ has the sup norm then the norm of $p \in C(\sigma(T))$ is $\|p\| = \sup \{ |p(\lambda)| \mid \lambda \in \sigma(T) \}$. By the Spectral Mapping Theorem (Theorem 8.5), $\mu = p(\lambda)$ for some $\lambda \in \sigma(T)$ if and only if $\mu \in \sigma(p(T))$, so we also have $\|p\| = \sup \{ |\mu| \mid \mu \in \sigma(p(T)) \} = r(p(T))$ (where $r(p(T))$ is the spectral radius of $p(r)$; the equality holds by the definition of spectral radius).
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Proposition 8.26 (continued 1)

**Proof (continued).** Since $T$ is a normal operator, by Theorem 8.23, $\|p\| = r(p(T)) = \|p(T)\|$. Therefore the mapping of $p$ to $p(T)$ is an isometry.

The set of polynomials in $C(\sigma(T))$ form a dense subset of $C(\sigma(T))$ by the Stone-Weierstrass Theorem (see Appendix B of Promislow or Section 12.3 of Royden and Fitzpatrick). By Theorem 2.20, the mapping of the set of polynomials in $C(r(T))$ to $B(H)$ (notice $B(H)$ is a Banach space by Theorem 2.15) can be extended to an isometry defined on all of $X(\sigma(T))$. We now prove each of the four claims.
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Let $f, g \in C(\sigma(T))$. Then, since the set of polynomials is dense in $C(\sigma(T))$ then there are sequence of polynomials $(p_n)$ and $(q_n)$ with $(p_n) \to f$ and $(q_n) \to g$ (the convergence is uniform since $C(\sigma(T))$ is equipped with the sup norm).
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Proof (continued). (i) Now $p_n + q_n = f + g$ and so

$$(f + g)(T) = \left( \lim_{n \to \infty} (p_n + q_n)(T) \right) = \lim_{n \to \infty} (p_n(T) + q_n(T))$$

$$= \lim_{n \to \infty} p_n(T) + \lim_{n \to \infty} q_n(T) = f(T) + g(T).$$

For $\alpha \in \mathbb{C}$,

$$(\alpha f)(T) = \lim_{n \to \infty} (\alpha p_n)(T) = \lim_{n \to \infty} \alpha p_n(T) = \alpha \lim_{n \to \infty} p_n(T) = \alpha f(T).$$
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(ii) We have

$$\bar{f}(T) = \left( \lim_{n \to \infty} p_n \right)(T) \lim_{n \to \infty} (\overline{p_n}(T))$$

$$= \lim_{n \to \infty} (\overline{p_n}(T^*)) \text{ since } T = T^*$$

$$= \lim_{n \to \infty} (p_n(T))^* \text{ by Theorem 4.26 (consider } p_n \text{ in terms of its coefficients and the fact that } T = T^*$$

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Proof (continued). (iii) We have

\[(fg)(T) = \lim_{n \to \infty} (p_n q_n)(T) = \lim_{n \to \infty} (p_n(T)q_n(T))\]

\[= \lim_{n \to \infty} p_n(T) \lim_{n \to \infty} q_n(T) = f(T)g(T).\]

(iv) Let \(X \in \mathcal{B}(H)\) such that \(ST = TS\). Notice that

\[S(a_i T^i + a_j T^j) = a_i S(T^i) + a_j S(T^j) = a_i T^i S + a_j T^j S = (a_i T^i + a_j T^j)S.\]

So for a polynomial \(p\), \(Sp(T) = p(T)S\).
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\[Sf(T) = S\left(\lim_{n \to \infty} p_n(T)\right)\]

\[= \lim_{n \to \infty} Sp_n(T) \text{ since } S \text{ is continuous}\]

\[= \lim_{n \to \infty} (p_n(T)S) \text{ as argued above}\]

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