Introduction to Functional Analysis

Chapter 9. Compact Operators

9.1. Introduction and Basic Definitions—Proofs of Theorems

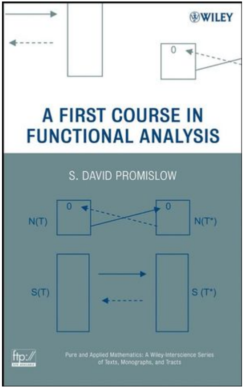

Table of contents

Lemma 9.1.A

Lemma 9.1.A. A subset K of a normed linear space is relatively compact if and only if any sequence in K has a convergent subsequence (where the limit if an element of the normed linear space, but not necessarily in K).

Proof. Suppose K is relatively compact and let $(x_n) \subset K$. Then $(x_n) \subset \overline{K}$ and $\overline{K}$ is compact. By the second definition of "compact set" (see the class notes for Section 2.2.), (x_n) has a subsequence (x_{n_k}) which converges to a point in $\overline{K}$.

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Suppose every sequence in K has a convergent subsequence. Consider a sequence $(x_n) \subset \overline{K}$. If finitely many of the x_n lie in K, then by hypothesis there is a subsequence of (x_n) which converges. The limit of the subsequence must be in $\overline{K}$ by part (iv) of the definition of "closure" of a set (see Section 2.2).

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Lemma 9.1.A (continued)

Proof (continued). If (x_n) contains only finitely many terms in K, we can assume without loss of generality that $(x_n) \subset \overline{K} \setminus K$. Let $\varepsilon > 0$. Since each element of $\overline{K} \setminus K$ is the limit of a sequence of points in K (by part (iv) of the definition of closure) then for each $x_n \in \overline{K} \setminus K$ there is $x_n \in K$ such that $\|x_n - x_n'\| < \varepsilon/2^n$. Then $(x_n') \subset K$ and so by hypothesis there is a subsequence (x_{n_k}') which converges to some $x \in \overline{K}$. So there is $N \in \mathbb{N}$ such that for all $k \geq N$ we have $|x_{n_k}' - x| < \varepsilon/2$.

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$$\|x_{n_k} - x\| = \|x_{n_k} - x'_{n_k} + x'_{n_k} - x\| \leq \|x_{n_k} - x'_{n_k}\| + \|x'_{n_k} - x\| < \varepsilon/2^{n_k} + \varepsilon/2 < \varepsilon.$$

Therefore $(x_{n_k}) \to x \in \overline{K}$. So every sequence of elements in $\overline{K}$ has a subsequence which converges to an element of $\overline{K}$. Therefore, by the second definition of "compact set," $\overline{K}$ is compact. □

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Lemma 9.1.B

Lemma 9.1.B. Let X and Y be normed linear spaces. Then $T \in \mathcal{B}(X, Y)$ is a compact operator if an only if given any bounded sequence (x_n) in X, the sequence (Tx_n) has a convergent subsequence.

Proof. Suppose T is compact. Let (x_n) be a bounded sequence in X and let $B = \{x_n \mid n \in \mathbb{N}\}$. Since T is compact then $T(B)$ is relatively compact and $(Tx_n) \subset T(B)$. By Lemma 9.1.A, (Tx_n) has a convergent subsequence.

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Suppose that for every bounded sequence (x_n) in X, the sequence (Tx_n) has a convergent subsequence. Let B be a founded set, and consider $T(B)$. Let (y_n) be a sequence in $T(B)$. Then there are $x_n \in B$ such that $Tx_n = y_n$. So (x_n) is a bounded sequence in X.

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