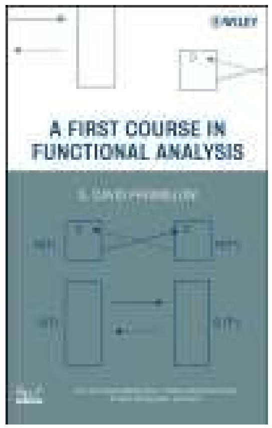
# Introduction to Functional Analysis

**Chapter 9. Compact Operators**

9.2. Compactness Criteria in Metric Spaces—Proofs of Theorems








## Proposition 9.1

**Proposition 9.1.** Set $A$ in a metric space is totally bounded if and only if any sequence $(a_n)$ of points in $A$ has a Cauchy subsequence.

**Proof.** First, suppose that $(a_n)$ is a sequence in totally bounded set $A$. Notice that for any given $\varepsilon > 0$, there is a $\varepsilon/2$-net $F$ for $A$ since $A$ is totally bounded. Since $F$ contains only finitely many points but the sequence $(a_n)$ contains infinitely many (not necessarily distinct) terms, then from some $y \in F$ we must have that $d(a_n, y) \le \varepsilon/2$ for infinitely many $a)n$. These $a_n$ form a subsequence of $(a_n)$ in which any two terms are within $\varepsilon$ of each other.

With $\varepsilon_1 = 1$, create subsequence $(a_n^1)$ of $(a_n)$ as described in the previous paragraph. Then iterate this process creating sequence $(a_n^{k+1})$ a subsequence of $a_n^k)$ using $\varepsilon = 1/(k+1)$ (so for each $k \in \mathbb{N}$, all terms in $(a_n^{k+1})$ are within $\varepsilon = 1/k$ of each other). Now define sequence $(a_n^n)$ and let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ with $1/N < \varepsilon$.





## Proposition 9.1 (continued)

**Proof (continued).** The tail of the sequence $(a_n^n)_{n=N}^\infty$ is a subsequence of $(a_n^N)$ and so the terms of the tail are within $1/N < \varepsilon$ of each other. That is, for all $n, m \ge N$ we have $\|a_n^n - a_m^m\| \le 1/N < \varepsilon$ and $(a_n^n)$ is a Cauchy subsequence of $(a_n)$.

We prove the contrapositive of the converse. Suppose $A$ is not totally bounded. Then for some $\varepsilon_0 > 0$ such that no $\varepsilon_0$-net for $A$ exists. Let $a_1 \in A$. Choose $a_2 \in A$ with $d(a_1, a_2) > \varepsilon_0$ (which can be done since $F = \{a_1\}$ is not a $\varepsilon_0$-net). Suppose $a_1, a_2, \ldots, a_k$ have been chosen in $A$ where any two of these points are a distance more than $\varepsilon_0$ apart. Since $F = \{1_1, 1_2, \ldots, a_k\}$ is not a $\varepsilon_0$-net for $A$ then there is $a_{k+1} \in A$ such that $d(a_i, a_{k+1}) > \varepsilon_0$ for $i = 1, 2, \ldots, k$. So in the resulting sequence $(a_k)$, any pair of terms are a distance of at least $\varepsilon_0$ apart. So $(a_k)$ has no Cauchy subsequence. That is, if $A$ is not totally bounded then there is a sequence of points in $A$ with no Cauchy subsequence. □





## Proposition 9.2

**Proposition 9.2.** A bounded set $A$ of a Banach space $X$ is relatively compact if and only if for any $\varepsilon > 0$ there is a finite dimensional subspace $Y$ of $X$ with $A \subseteq^\varepsilon Y$.

**Proof.** Suppose $A$ has the finite dimensional subspace property. Given $\varepsilon > 0$, choose a finite dimensional subspace $Y$ such that $A \subseteq^{\varepsilon/2} Y$. For each $a \in A$ choose $a' \in Y$ such that $\|a - a'\| \le \varepsilon/2$. The set $A'$ consisting of all such $a'$ is bounded since $\text{diam}(A') \le \text{diam}(A) + \varepsilon$. Notice that, by construction, $A \subseteq^{\varepsilon/2} A'$. Since $Y$ is finite dimensional, then it is closed by Theorem 2.31(c). So $\overline{A}' \subset \overline{Y}$. Now $\overline{A}'$ is a closed and bounded set in finite dimensional linear space $Y$, so by the Heine Borel Theorem, $\overline{A}'$ is compact. So set $A'$ is relatively compact in $Y$. So $A'$ is totally bounded, by Corollary 9.2.A, and there is a finite set $F \subset Y$ where $A' \subseteq^{\varepsilon/2} F$. Therefore $A \subseteq^\varepsilon F$, so $A$ is totally bounded. Since $X$ is complete (a Banach space) then $A$ is relatively compact by Corollary 9.2.A.

# Proposition 9.2 (continued)

**Proposition 9.2.** A bounded set $A$ of a Banach space $X$ is relatively compact if and only if for any $\varepsilon > 0$ there is a finite dimensional subspace $Y$ of $X$ with $A \subseteq^{\varepsilon} Y$.

**Proof (continued).** For the converse, suppose $A$ is relatively compact. By Corollary 9.2.A, set $A$ is totally bounded and so, by definition, for given $\varepsilon > 0$ there is finite set $F \subset X$ such that $A \subseteq^{\varepsilon} F$. Then $Y = \text{span}(F)$ is finite dimensional and $A \subseteq^{\varepsilon} Y$. □





# Proposition 9.3

**Proposition 9.3.** Let $S$ be a set and $B(S)$ the set of functions from $S$ to field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) under the sup norm. Suppose that $A$ is a bounded subset of $B(E)$ satisfying the following: For any $\varepsilon > 0$, we can partition $S$ into a finite number of pairwise disjoint subsets $S_1, S_2, \ldots, S_n$ such that, given any $i$, any two points $s, t \in S_i$, and any $f \in A$, we have $|f(s) - f(t)| \le \varepsilon$. Then $A$ is relatively compact (in $B(S)$).

**Proof.** First, $B(S)$ is a Banach space by Theorem 2.14. Let $\varepsilon > 0$ be given and let the partition of $S$ be $S_1, S_2, \ldots, S_n$. Let $Y$ be the subspace of $B(S)$ consisting of all functions that are constant on each $S_i$. Then $Y$ is finite dimensional with basis $\{1_{S_i} \mid i = 1, 2, \ldots, n\}$ (where $1_{S_i}$ is the constant function 1 on $S_i$ and 0 elsewhere). For any $f \in A$, choose $s_i \in S_i$ for $i = 1, 2, \ldots, n$ and let $g$ be the element of $Y$ that takes the value $f(s_i)$ on $S_i$ (so $g$ is defined on all of $S$ since the $S_i$ form a partition of $S$).





# Proposition 9.3 (continued)

**Proposition 9.3.** Let $S$ be a set and $B(S)$ the set of functions from $S$ to field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) under the sup norm. Suppose that $A$ is a bounded subset of $B(E)$ satisfying the following: For any $\varepsilon > 0$, we can partition $S$ into a finite number of pairwise disjoint subsets $S_1, S_2, \ldots, S_n$ such that, given any $i$, any two points $s, t \in S_i$, and any $f \in A$, we have $|f(s) - f(t)| \le \varepsilon$. Then $A$ is relatively compact (in $B(S)$).

**Proof (continued).** For $t \in S_i$ we have

$$|f(s_i) - f(t)| = |g(s_i) - f(t)| = |g(t) - f(t)| \le \varepsilon$$

by hypothesis. So for any $t \in S$, $|g(t) - f(t)| \le \varepsilon$. Since $f \in A$ is arbitrary, we have shown that $A \subseteq^{\varepsilon} Y$. Since $X = B(S)$ is a Banach space, $Y$ is finite dimensional, and $A \subseteq^{\varepsilon} Y$ for any given $\varepsilon > 0$, then by Proposition 9.2 $A$ is relatively compact (in $X = B(S)$). □





# Theorem 9.5. Arzela-Ascoli Theorem

**Theorem 9.5. Arzela-Ascoli Theorem.**
If $S$ is a compact metric space, a subset $A$ of $C(S)$ (the set of continuous real valued or complex valued functionals on $S$) is relatively compact if and only if it is bounded and equicontinuous.

**Proof of the "if" part.** Suppose $A \subset C(S)$ is bounded and equicontinuous. Since $S$ is compact, then for any $f \in C(S)$ we have that $f(S)$ is compact and so $f(S)$ is bounded by the Compact Set Theorem (see the class notes for Section 2.2). So $C(S)$ is a subspace of $B(S)$. We now use Proposition 9.3 to show that $A$ is relatively compact. Let $\varepsilon > 0$. Then by the equicontinuity of set $A$, there is $\delta > 0$ such that $|s - t| < \delta$ implies that $|f(s) - f(t)| < \varepsilon$ for all $f \in A$. Since $S$ is compact then it has a $\delta$-net (cover $S$ with all $\delta$ radius balls and then choose a finite subcover; the centers of the resulting finite number of balls form a $\delta$-net), say $\{t_1, t_2, \ldots, t_n\}$.

## Theorem 9.5. Arzela-Ascoli Theorem (continued)

**Theorem 9.5. Arzela-Ascoli Theorem.**
If $S$ is a compact metric space, a subset $A$ of $C(S)$ (the set of continuous real valued or complex valued functionals on $S$) is relatively compact if and only if it is bounded and equicontinuous.

**Proof of the "if" part (continued).** Then partition $S$ as follows: Let $S_1 = \{s \in S \mid d(x, t_1) < \delta\}$ and define inductively

$$S_k = s \in S \mid s \notin \cup_{i=1}^{k-1} S_i \text{ and } d(s, t_k) < \delta\}.$$

Since the $t_i$'s form a $\delta$-net for $S$, the union of all the sets $S_i$ equals $S$. By construction, the $S_i$'s are pairwise disjoint, and so partition $S$. So the hypotheses of Proposition 9.3 are satisfied and hence $A$ is relatively compact. □





## Theorem 9.6

**Theorem 9.6.** Let $A$ be a bounded subset of $\ell^p$ that has uniformly small tails. That is, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $f \in A$, $\sum_{i=N}^{\infty} |f(i)|^p < \varepsilon$. Then $A$ is relatively compact.

**Proof.** Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} |f(i)| < \varepsilon^p$. Let $Y$ be the set of all $f$ such that $f(k) = 0$ for $k \geq N$. Then, since $N$ is now fixed, $Y$ is a finite dimensional subspace with basis $\{\delta_i \mid i = 1, 2, \ldots, N-1\}$ (where $\delta_i$ is the $i$the standard basis vector). For any $f \in A$, consider $f' \in Y$ such that $f'$ agrees with $f$ on integers less than $N$ (that is, $f(i) = f'(i)$ for $i = 1, 2, \ldots, N-1$). Then $f - f'$ consists of $N-1$ 0's followed by the "tail" of $f$. So

$$\|f - f'\|_p = \left\{\sum_{i=N}^{\infty} |f(i)|^p\right\}^{1/p} < \varepsilon^{p1/p} = \varepsilon.$$

Therefore $A \subseteq^{\varepsilon} Y$. By Proposition 9.2, $A$ is relatively compact. □





## Theorem 9.7

**Theorem 9.7.** The multiplication operator $M_f$ on $\ell^p$ is compact if and only if $f(n) \to 0$.

**Proof.** Recall that the multiplication operator $M_f$ for $f \in \ell^p$ is defined as $M_f(g) = (f(n)g(n))_{n=1}^{\infty}$. Suppose $f(n) \to 0$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $|f(n)| < \varepsilon^{1/p}$ for $n \geq N$. Let $g \in B(1) = \{g \in \ell^p \mid \|g\| < 1\}$. Then

$$\sum_{i=N}^{\infty} |f(i)g(i)|^p = \sum_{i=N}^{\infty} |f(i)|^p |g(i)|^p \leq \varepsilon \sum_{i=N}^{\infty} |g(i)|^p \leq \varepsilon \|g\|_p^p < \varepsilon.$$

So the set $M_f(B(1))$ has uniformly small tails. By Proposition 9.6, $M_f(B(1))$ is relatively compact. Hence (by definition and the observation in the previous notes) $M_f$ is a compact operator.





## Theorem 9.7 (continued)

**Proof (continued).** We consider the contrapositive of the converse. Suppose $f(n)$ does not converge to 0. Then there is $\varepsilon_0 > 0$ and an infinite set $J \in \mathbb{N}$ such that $|f(n)| \geq \varepsilon_0$ for all $n \in J$. Now $B(1)$ is the open unit ball in $\ell^p$, so each standard basis vector $S_i$ satisfies $\delta_i/2 \in B(1)$. So $M_f(\delta_i/2) \in M_f(B(1))$ and hence $M_f(B(1))$ contains the points $(|f(n)|/2)\delta_n$ for all $n \in J$. But the $\ell^p$ distance between any two such points satisfies

$$\|(f(n)/2)\delta_n - (f(m)/2)\delta_m\|_p = \{|f(n)/2|^p + |f(m)/2|^p\}^{1/p}$$

$$\geq (\varepsilon_0^p/2^p + \varepsilon_0^p/2^p)^{1/p} = 2^{1/p-1}\varepsilon > \varepsilon/2.$$

That is, $M_f(B(1))$ contains an infinite set of points, any pair of which are a distance of at least $\varepsilon_0/2$ apart. Treating this infinite set as a sequence (namely, $((f(n)/2)\delta_n)_{n\in J}$) it has no Cauchy subsequence, so by Proposition 9.2 $M_f(B(1))$ is not totally bounded. Hence (by definition and the observation in the previous not) $M_f$ is not a compact operator. □

# Theorem 9.9

**Theorem 9.9.** The operator $K$ given in Example 9.8 if compact.

**Proof.** Since $K$ is bounded then the set $K(B(1))$ is bounded. We now show that $K(B(1))$ is equicontinuous. Since $k$ is continuous and the closed unit square is compact, then $k$ is uniformly continuous. So for give $\varepsilon > 0$ there is $\delta > 0$ such that if $|s_1 - s_2| < \delta$, where $s_1, s_2 \in [0, 1]$, then $|k(s_1, t) - k(s_2, t)| < \varepsilon$ for all $t \in [0, 1]$. So for all $f \in B(1)$, if $|s_1 - s_2| < \delta$ then

$$|K(f(s_1)) - K(f(s_2))| \leq \int_0^1 |k(s_1, t) - k(s_2, t)||f(t)|\, dt \leq \varepsilon \int_0^1 |f(t)|\, dt < \varepsilon$$

(since $f \in B(1)$). Therefore the set $K(B(1))$ is equicontinuous. By the Azrela-Ascoli Theorem (Theorem 9.5), $M_f(B(1))$ is relatively compact. Hence (by definition and the observation in the note above) $M_f$ is a compact operator. □