Proposition 9.1. Let $\epsilon > 0$. Then there is a natural number $N \in \mathbb{N}$ with $I/N < \epsilon$ and $N/N < \epsilon$. Now define a sequence $(a_n)$ and $(b_n)$ by $a_n = (I/n)$ for each $n \in \mathbb{N}$ and $b_n = (I/n)$ for each $n \in \mathbb{N}$. Let $c = (a_n + b_n)$. Proceed as described in the previous paragraph. Then iterate this process creating a subsequence of $(c_n)$. Let $d = (c_1, c_2, \ldots)$. Then $(d_n)$ is a Cauchy sequence.

Proposition 9.2. If a Cauchy sequence is bounded, it is uniformly Cauchy.

Proposition 9.3. If a Cauchy sequence is bounded and Cauchy, it converges.

Definition. A complete metric space is a metric space in which every Cauchy sequence converges.

Theorem. Every complete metric space is both bounded and totally bounded.

Proof. Suppose $X$ is a complete metric space. Then $X$ is compact, and therefore every sequence in $X$ has a convergent subsequence. Since $X$ is complete, every Cauchy sequence converges. Therefore, every sequence in $X$ converges. This implies that $X$ is compact.

Corollary. Every closed subset of a complete metric space is itself complete.

Proof. Suppose $A$ is a closed subset of a complete metric space $X$. If $(x_n)$ is a sequence in $A$, then $(x_n)$ converges to some $x \in X$. Since $A$ is closed, $x \in A$. Therefore, every convergent sequence in $A$ converges to a point in $A$. This implies that $A$ is complete.

Chapter 4. Compactness Criteria in Metric Spaces—Proofs of Theorems
Proposition 9.2. A set of functions from a finite dimensional and a closed subspace $X$, there is a finite set of pairwise disjoint subsets of $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

By Corollary 9.2, $Y$ is a closed subspace of $X$. For any $y \in X$, we can find a closed subspace $Z$ of $X$ such that $y \in Z$. Then $Z = \text{span}(y)$.

For the converse, suppose $X$ is a finite dimensional and a closed subspace $Y$. Then $Y = \text{span}(y)$.

Proposition 9.3. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proposition 9.4. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proposition 9.5. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Theorem 9.5. Arzelà-Ascoli Theorem.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proposition 9.6. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proposition 9.7. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.

Proposition 9.8. A set of functions from a finite dimensional and a closed subspace $X$. For any $y \in X$, we can find a closed subspace $Y$ of $X$ such that $y \in Y$. Then $Y = \text{span}(y)$.

Proof. First, $Y$ is a closed subspace of $X$. Let $y \in Y$. Then $y = \text{span}(y)$.
Theorem 9.7 (continued)

Therefore $A \subseteq Y$. By Proposition 9.2, $A$ is relatively compact.

$$\varepsilon = \frac{1}{2} \left( \left\| \frac{d(f)}{d} \right\| \right) = \frac{1}{2} \left( \left\| \frac{d(f)}{d} \right\| \right)$$

So the hypotheses of Proposition 9.2 are satisfied, and hence $A$ is relatively compact.

Theorem 9.6, let $A$ be a bounded subset of $\mathbb{C}$ that is uniformly small.

Theorem 9.5, Artaza-Archbald Theorem (continued)

If $F$ is a compact metric space, then the set of continuous functions $\mathbb{C}(F)$ is relatively compact if $F$ is a compact metric space with countable subbasis, and only if it is bounded and equicontinuous.
compact operator.

Hence (by definition and the observation in the note above) \( M_f \) is a compact operator. Therefore the set \( K(B(1)) \) is equicontinuous. By the Arzela-Ascoli Theorem (Theorem 9.5), \( M_f(B(1)) \) is relatively compact. Therefore the set \( K(B(1)) \) is equicontinuous.

\( \therefore \exists \epsilon > 0 \text{ s.t. } \int_0^\infty \frac{1}{t} \left| K_{x_1}(t) - K_{x_2}(t) \right| dt < \delta \). Since \( K(B(1)) \) is equicontinuous, for all \( f \in B(1) \), the closed unit sphere is compact, then \( K \) is uniformly continuous. So for \( \delta \) small, there is a \( \epsilon > 0 \) such that \( \left| K_{x_1}(t) - K_{x_2}(t) \right| dt < \delta \) whenever \( |t_1 - t_2| \). So for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \left| K_{x_1}(t) - K_{x_2}(t) \right| dt < \delta \) whenever \( |t_1 - t_2| \). Therefore, since \( K(B(1)) \) is equicontinuous, the operator \( K \) is continuous. Since \( K \) is continuous and the ball \( B(1) \) is closed, then the set \( K(B(1)) \) is bounded. We now show that \( K(B(1)) \) is equicontinuous. Since \( K \) is continuous and the ball \( B(1) \) is closed, then the set \( K(B(1)) \) is bounded.

**Theorem 9.9.** The operator \( K \) is given in Example 9.8. It is compact.