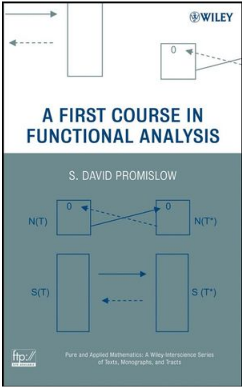
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.2. Compactness Criteria in Metric Spaces—Proofs of Theorems



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With $\varepsilon_1 = 1$, create subsequence (a_n^1) of (a_n) as described in the previous paragraph. Then iterate this process creating sequence (a_n^{k+1}) a subsequence of $a_n^k)$ using $\varepsilon = 1/(k+1)$ (so for each $k \in \mathbb{N}$, all terms in (a_n^{k+1}) are within $\varepsilon = 1/k$ of each other). Now define sequence (a_n^n) and let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ with $1/N < \varepsilon$.

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Proposition 9.1 (continued)

Proof (continued). The tail of the sequence $(a_n^n)_{n=N}^{\infty}$ is a subsequence of (a_n^N) and so the terms of the tail are within $1/N < \varepsilon$ of each other. That is, for all $n, m \geq N$ we have $\|a_n^n - a_m^m\| \leq 1/N < \varepsilon$ and (a_n^n) is a Cauchy subsequence of (a_n).

We prove the contrapositive of the converse. Suppose A is not totally bounded. Then for some $\varepsilon_0 > 0$ such that no ε_0-net for A exists.

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Proposition 9.2

Proposition 9.2. A bounded set A of a Banach space X is relatively compact if and only if for any $\varepsilon > 0$ there is a finite dimensional subspace Y of X with $A \subseteq^{\varepsilon} Y$.

Proof. Suppose A has the finite dimensional subspace property. Given $\varepsilon > 0$, choose a finite dimensional subspace Y such that $A \subseteq^{\varepsilon/2} Y$. For each $a \in A$ choose $a' \in Y$ such that $\|a - a'\| \leq \varepsilon/2$. The set A' consisting of all such a' is bounded since $\mathrm{diam}(A') \leq \mathrm{diam}(A) + \varepsilon$. Notice that, by construction, $A \subseteq^{\varepsilon/2} A'$.

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Proof (continued). For the converse, suppose A is relatively compact. By Corollary 9.2.A, set A is totally bounded and so, by definition, for given $\varepsilon > 0$ there is finite set $F \subset X$ such that $A \subseteq^{\varepsilon} F$. Then $Y = \text{span}(F)$ is finite dimensional and $A \subseteq^{\varepsilon} Y$. □

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Proposition 9.3. Let S be a set and $B(S)$ the set of functions from S to field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) under the sup norm. Suppose that A is a bounded subset of $B(E)$ satisfying the following: For any $\varepsilon > 0$, we can partition S into a finite number of pairwise disjoint subsets $S_1, S_2, \ldots, S_n$ such that, given any i, any two points $s, t \in S_i$, and any $f \in A$, we have $|f(s) - f(t)| \leq \varepsilon$. Then A is relatively compact (in $B(S)$).

Proof. First, $B(S)$ is a Banach space by Theorem 2.14. Let $\varepsilon > 0$ be given and let the partition of S be $S_1, S_2, \ldots, S_n$. Let Y be the subspace of $B(S)$ consisting of all functions that are constant on each S_i.

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Theorem 9.5. Arzela-Ascoli Theorem

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If S is a compact metric space, a subset A of $C(S)$ (the set of continuous real valued or complex valued functionals on S) is relatively compact if and only if it is bounded and equicontinuous.

Proof of the "if" part. Suppose $A \subset C(S)$ is bounded and equicontinuous. Since S is compact, then for any $f \in C(S)$ we have that $f(S)$ is compact and so $f(S)$ is bounded by the Compact Set Theorem (see the class notes for Section 2.2). So $C(S)$ is a subspace of $B(S)$.

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$$S_k = s \in S \mid s \notin \cup_{i=1}^{k-1} S_i \text{ and } d(s, t_k) < \delta\}.$$

Since the t_i's form a δ-net for S, the union of all the sets S_i equals S. By construction, the S_i's are pairwise disjoint, and so partition S. So the hypotheses of Proposition 9.3 are satisfied and hence A is relatively compact. □

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Theorem 9.6

Theorem 9.6. Let A be a bounded subset of ℓ^p that has uniformly small tails. That is, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $f \in A$, $\sum_{i=N}^{\infty} |f(i)|^p < \varepsilon$. Then A is relatively compact.

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$$\|f - f'\|_p = \left\{ \sum_{i=N}^{\infty} |f(i)|^p \right\}^{1/p} < \varepsilon^{p1/p} = \varepsilon.$$

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$$\|f - f'\|_p = \left\{ \sum_{i=N}^{\infty} |f(i)|^p \right\}^{1/p} < \varepsilon^{p1/p} = \varepsilon.$$

Therefore $A \subseteq^{\varepsilon} Y$. By Proposition 9.2, A is relatively compact. □

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Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} |f(i)| < \varepsilon^p$. Let Y be the set of all f such that $f(k) = 0$ for $k \geq N$. Then, since N is now fixed, Y is a finite dimensional subspace with basis $\{\delta_i \mid i = 1, 2, \ldots, N-1\}$ (where δ_i is the $i$the standard basis vector). For any $f \in A$, consider $f' \in Y$ such that f' agrees with f on integers less than N (that is, $f(i) = f'(i)$ for $i = 1, 2, \ldots, N-1$). Then $f - f'$ consists of $N-1$ 0's followed by the "tail" of f. So

$$\|f - f'\|_p = \left\{ \sum_{i=N}^{\infty} |f(i)|^p \right\}^{1/p} < \varepsilon^{p1/p} = \varepsilon.$$

Therefore $A \subseteq^{\varepsilon} Y$. By Proposition 9.2, A is relatively compact. $\square$

Theorem 9.7

Theorem 9.7. The multiplication operator M_f on ℓ^p is compact if and only if $f(n) \to 0$.

Proof. Recall that the multiplication operator M_f for $f \in \ell^p$ is defined as $M_f(g) = (f(n)g(n))_{n=1}^{\infty}$. Suppose $f(n) \to 0$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $|f(n)| < \varepsilon^{1/p}$ for $n \geq N$.

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$$\sum_{i=N}^{\infty} |f(i)g(i)|^p = \sum_{i=N}^{\infty} |f(i)|^p |g(i)|^p \leq \varepsilon \sum_{i=N}^{\infty} |g(i)|^p \leq \varepsilon \|g\|_p^p < \varepsilon.$$

So the set $M_f(B(1))$ has uniformly small tails.

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So the set $M_f(B(1))$ has uniformly small tails. By Proposition 9.6, $M_f(B(1))$ is relatively compact. Hence (by definition and the observation in the previous notes) M_f is a compact operator.

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Theorem 9.7 (continued)

Proof (continued). We consider the contrapositive of the converse. Suppose $f(n)$ does not converge to 0. Then there is $\varepsilon_0 > 0$ and an infinite set $J \in \mathbb{N}$ such that $|f(n)| \geq \varepsilon_0$ for all $n \in J$. Now $B(1)$ is the open unit ball in ℓ^p, so each standard basis vector S_i satisfies $\delta_i/2 \in B(1)$. So $M_f(\delta_i/2) \in M_f(B(1))$ and hence $M_f(B(1))$ contains the points $(|f(n)|/2)\delta_n$ for all $n \in J$. But the ℓ^p distance between any two such points satisfies

$$\|(f(n)/2)\delta_n - (f(m)/2)\delta_m\|_p = \{|f(n)/2|^p + |f(m)/2|^p\}^{1/p}$$

$$\geq (\varepsilon_0^p/2^p + \varepsilon_0^p/2^p)^{1/p} = 2^{1/p-1}\varepsilon > \varepsilon/2.$$

That is, $M_f(B(1))$ contains an infinite set of points, any pair of which are a distance of at least $\varepsilon_0/2$ apart.

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Theorem 9.9. The operator K given in Example 9.8 if compact.

Proof. Since K is bounded then the set $K(B(1))$ is bounded. We now show that $K(B(1))$ is equicontinuous.

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$$|K(f(s_1)) - K(f(s_2))| \leq \int_0^1 |k(s_1, t) - k(s_2, t)||f(t)|\, dt \leq \varepsilon \int_0^1 |f(t)|\, dt < \varepsilon$$

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