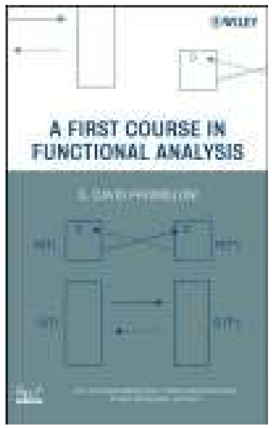
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.3. New Compact Operators from Old—Proofs of Theorems



Proposition 9.10

Proposition 9.10. Compactness and Algebraic Properties.
Suppose we are given normed linear spaces X, Y, and Z, a scalar α, operators $S, T \in \mathcal{B}(X, Y)$, and $A \in \mathcal{B}(Y, Z)$. We have:

(a) If T is compact, then so is αT.
(b) If S and T are compact, then so is $S + T$.
(c) If T is compact, then so is $A \circ T = AT$.
(d) If A is compact, then so is $A \circ T = AT$.

Proof. (a) Multiplying a subset by a scalar preserves relative compactness. So if $T(B(1))$ is relatively compact then so is $\alpha T(B(1))$ and the compactness of T implies the compactness of αT.

(b) If T is a compact operator, then by the definition of "compact operator," for any sequence (x_n) in $B(1)$, the sequence $(T(x_n)$ has a convergent subsequence, say (x_n') in $B(1)$. Similarly, the sequence $(S(x_n'))$ has a convergent subsequence x_n''). So the sequence $((S + T)(x_n'')$ converges and by the definition of "compact operator," $S + T$ is compact.

Proposition 9.10 (continued)

Proposition 9.10. Compactness and Algebraic Properties.
Suppose we are given normed linear spaces X, Y, and Z, a scalar α, operators $S, T \in \mathcal{B}(X, Y)$, and $A \in \mathcal{B}(Y, Z)$. We have:

(a) If T is compact, then so is αT.
(b) If S and T are compact, then so is $S + T$.
(c) If T is compact, then so is $A \circ T = AT$.
(d) If A is compact, then so is $A \circ T = AT$.

Proof (continued). (c) If T is a compact operator, then by the definition of "compact operator," for any sequence (x_0) in $B(1)$, the sequence $(T(x_n))$ has a convergent subsequence, say $(T(x_n'))$. Since A is continuous then $((S \circ T)(x_n')) = (ST(x_n'))$ converges. So ST is compact.

(d) Since T is bounded, then $T(B(1))$ is bounded. Since A is compact, then by the definition of "compact operator," $A(T(B(1))$ is relatively compact. So (again, by definition) $A \circ T = AT$ is compact. □

Proposition 9.11

Proposition 9.11. Compactness and Limits.
If $T = \lim T_n$, in which (T_n) is a sequence of compact operators in $\mathcal{B}(X, Y)$, then T is compact.

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|T - T_N\| < \varepsilon/2$. Since T_N is a compact operator, then (by the definition and the note on page 187) $T(B(1))$ is relatively compact. So by Proposition 9.2 there is finite dimensional $Y \subset Z$ such that $Y_N(B(1)) \subseteq^{\varepsilon/2} Y$. Now for any $x \in B(1)$, $\|T(x) - T_N(x)\|_Z = \|(T - T_N)(x)\|_Z \le \|T - T_N\|\|x\|_X < \varepsilon/2$, so $T(B(1)) \subseteq^{\varepsilon/2} T_N(B(1))$. Therefore, $T(B(1)) \subseteq^{\varepsilon} Y$, so $T(B(1))$ is relatively compact by Proposition 9.2. That is, T is (by definition and page 187) compact. □

Proposition 9.12

Proposition 9.12. Compactness and Adjoints.
Let $T \in \mathcal{B}(X, Y)$. If T is compact, then its adjoint T^* is compact. If Y is complete, then compactness of T^* implies that of T.

Proof. Suppose T is compact. Let (g_n) be a sequence in $B^*(1)$ (the unit ball of Y^*). Now $B^*(1)$ is a set of bounded functionals defined on all of Y (and so these functionals are continuous). For $B(1) \subset X$ we have $\overline{T(B(1))} \subset Y$, so we can restrict each element of $B^*(1)$ to the compact set $T(B(1))$ (since T is compact, $T(B(1))$ is relatively compact and so [by the definition of "relatively compact"] $\overline{T(B(1))}$ is compact). By Example 9.4, $B^*(1)$ (restricted to $\overline{T(B(1))}$) is equicontinuous. So $B^*(1)$ (restricted) is bounded and equicontinuous, and hence by the Arzela-Ascoli Theorem (Theorem 9.5) (with $S = \overline{T(B(1))}$ and $A \subset C(\overline{T(B(1))})$ as the set of elements of $B^*(1)$ restricted to $\overline{T(B(1))}$) $B^*(1)$ (restricted) is relatively compact.

Proposition 9.12 (continued 1)

Proposition 9.12. Compactness and Adjoints.
Let $T \in \mathcal{B}(X, Y)$. If T is compact, then its adjoint T^* is compact. If Y is complete, then compactness of T^* implies that of T.

Proof (continued). So (by definition and the note on page 187) $(g_n) \subset B^*(1)$ has a convergent subsequence (g_{n_k}) where $(g_{n_k}) \to g$ for some $g \in C(\overline{T(B(1))})$. Since $C(\overline{T(B(1))})$ has the sup norm (see Proposition 9.3), then g converges uniformly to g. So for given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \geq N$ and for $x \in B(1)$ we have $|g_{n_k}(Tx\_g(Tx)| < \varepsilon$ (notice $Tx \in T(B(1)) \subset \overline{T(B(1))}$). By the definition of adjoint (see Section 6.2), since $T : X \to Y$ and $g_{n_k}, g \in B^*(1) \subset Y^*$ then $(T^*g)(x) = g(T(Tx)$ and $T^*g_{n_k})(x) = g_{n_k}(Tx)$. Therefore $((T^*g_{n_k})(x) - (T^*g)(x)| < \varepsilon$. Hence $(T^*g_{n_k}) \to T^*g$. So for arbitrary sequence (g_n) in $B^*(1)$ the sequence (T^*g_n) has a convergent subsequence. So (by the note on page 187), T^* is compact.

Proposition 9.12 (continued 2)

Proposition 9.12. Compactness and Adjoints.
Let $T \in \mathcal{B}(X, Y)$. If T is compact, then its adjoint T^* is compact. If Y is complete, then compactness of T^* implies that of T.

Proof (continued). Conversely, suppose T^* is compact. By the previous paragraph, T^{**} is compact. Let j denote the natural embedding of X in X^{**} (see Theorem 6.8 where the notation would be $j(x) = \hat{x}$). By Exercise 6.5 we have $T^{**}(\hat{x}) = \widehat{T(x)}$, or $T^{**}(j(B(1)) = j(T(B(1))$. Since j is an isometry by Theorem 6.8, $j(B(1))$ is a subset of the unit ball in X^{**} and so is bounded. Since T^{**} is hypothesized to be compact the (by definition) $T^{**}(j(B(1)))$ is relatively compact. By Corollary 9.2.A, $T^{**}(j(B(1)))$ is therefore totally bounded. So $j(T(B(1)))$ is totally bounded. Since j is an isometry, then $T(B(1))$ is totally bounded. Y is hypothesized to be complete, so by Corollary 9.2.A, $T(B(1))$ is relatively compact. That is, by definition, T is compact. □