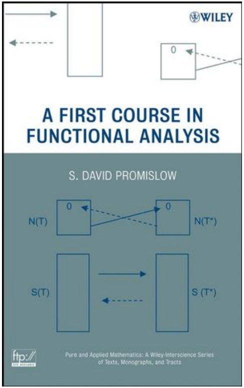
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.3. New Compact Operators from Old—Proofs of Theorems



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Proposition 9.10. Compactness and Algebraic Properties.
Suppose we are given normed linear spaces X, Y, and Z, a scalar α, operators $S, T \in \mathcal{B}(X, Y)$, and $A \in \mathcal{B}(Y, Z)$. We have:

(a) If T is compact, then so is αT.

(b) If S and T are compact, then so is $S + T$.

(c) If T is compact, then so is $A \circ T = AT$.

(d) If A is compact, then so is $A \circ T = AT$.

Proof. (a) Multiplying a subset by a scalar preserves relative compactness. So if $T(B(1))$ is relatively compact then so is $\alpha T(B(1))$ and the compactness of T implies the compactness of αT.

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(b) If T is a compact operator, then by the definition of "compact operator," for any sequence (x_n) in $B(1)$, the sequence $(T(x_n)$ has a convergent subsequence, say (x_n') in $B(1)$. Similarly, the sequence $(S(x_n'))$ has a convergent subsequence $x_n'')$. So the sequence $((S + T)(x_n'')$ converges and by the definition of "compact operator," $S + T$ is compact.

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Proof (continued). (c) If T is a compact operator, then by the definition of "compact operator," for any sequence (x_0) in $B(1)$, the sequence $(T(x_n))$ has a convergent subsequence, say $(T(x'_n))$. Since A is continuous then $((S \circ T)(x'_n)) = (ST(x'_n))$ converges. So ST is compact.

(d) Since T is bounded, then $T(B(1))$ is bounded. Since A is compact, then by the definition of "compact operator," $A(T(B(1))$ is relatively compact. So (again, by definition) $A \circ T = AT$ is compact. □

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(d) Since T is bounded, then $T(B(1))$ is bounded. Since A is compact, then by the definition of "compact operator," $A(T(B(1))$ is relatively compact. So (again, by definition) $A \circ T = AT$ is compact. □

Proposition 9.11

Proposition 9.11. Compactness and Limits.
If $T = \lim T_n$, in which (T_n) is a sequence of compact operators in $\mathcal{B}(X, Y)$, then T is compact.

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|T - T_N\| < \varepsilon/2$. Since T_N is a compact operator, then (by the definition and the note on page 187) $T(B(1))$ is relatively compact. So by Proposition 9.2 there is finite dimensional $Y \subset Z$ such that $Y_N(B(1)) \subseteq^{\varepsilon/2} Y$.

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Proposition 9.12

Proposition 9.12. Compactness and Adjoints.
Let $T \in \mathcal{B}(X, Y)$. If T is compact, then its adjoint T^* is compact. If Y is complete, then compactness of T^* implies that of T.

Proof. Suppose T is compact. Let (g_n) be a sequence in $B^*(1)$ (the unit ball of Y^*). Now $B^*(1)$ is a set of bounded functionals defined on all of Y (and so these functionals are continuous).

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Proof (continued). Conversely, suppose T^* is compact. By the previous paragraph, T^{**} is compact. Let j denote the natural embedding of X in X^{**} (see Theorem 6.8 where the notation would be $j(x) = \hat{x}$). By Exercise 6.5 we have $T^{**}(\hat{x}) = \widehat{T(x)}$, or $T^{**}(j(B(1)) = j(T(B(1))$. Since j is an isometry by Theorem 6.8, $j(B(1))$ is a subset of the unit ball in X^{**} and so is bounded. Since T^{**} is hypothesized to be compact the (by definition) $T^{**}(j(B(1)))$ is relatively compact.

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