Consider the resulting sequence of \( \langle y_n \rangle \) in the normed space \( X \). For any \( n \in \mathbb{N} \) and for all \( k \in \mathbb{Z} \), there is an index \( l \geq 1 \) such that \( y_{kn} \geq 0 \). Let \( Z = \{ y_n : n \in \mathbb{N} \} \). Then \( Z \) is a bounded above set in \( X \).

**Lemma 9.13**. Let \( Z \) be a bounded set. Suppose that \( Z \) is not onto. (b) Suppose that \( Z \) is onto.

**Proof**. (a) Suppose that \( Z \) is not onto (surjective). But since \( Z \) is one-to-one, its range is not onto. (b) Suppose that \( Z \) is onto. Let \( x \in X \). Then for some \( z \in Z \), \( x = f(z) \).

4.4 Spectrum of a Compact Operator—Problems of Theorems

**Chapter 4. Compact Operators**
Proposition 9.15 (continued)

First, we show that \( S \) is bounded below (see Section 3.4). Assume \( S \) is bounded below.

\[ \|z\| - \|x\| = \left\| \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right\| - \left\| \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} - \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2} \]

for \( a_i, b_i \geq 0 \) and \( a_i, b_i \leq 1 \) for all \( i \).

Then by Theorem 7.2, \( S \) is bounded below. 

Furthermore, let \( S \) and \( T \) be subspaces of \( \mathbb{R}^n \) such that \( S \subseteq T \). Then \( \|x\| \leq \|y\| \) for all \( x \in S \) and \( y \in T \).

Finally, we have that \( S \) is bounded below. 

Proposition 9.16

Let \( S \) be a Banach space and let \( T \subseteq S \) be a bounded linear operator. Then \( \|T\| = \sup \{ \|Tx\| : \|x\| \leq 1 \} \).

Corollary 9.4

Let \( x \in \mathbb{R}^n \) and let \( T \subseteq \mathbb{R}^n \) be a subset. Then \( \|x\| \leq \|T\| \) for all \( x \in T \).

Lemma 9.1

Theorem 9.5

Due to the boundedness of \( S \) as a subset of \( \mathbb{R}^n \), we have that \( S \) is bounded below. 

Again, note that these three subspaces are all compact.

Consider the sequence of scalars \( (\lambda_k) \subseteq \mathbb{R} \) where \( \lambda_k = \lambda \) for all \( k \in \mathbb{N} \).

Notice that these three subspaces are all compact.

Notice that these three subspaces are all compact.

Notice that these three subspaces are all compact.
Theorem 9.16. Let $T$ be a compact operator in $\mathcal{B}(X)$, in which $X$ is a Banach space. Then the nonzero elements of the spectrum of $T$ are all finite-dimensional, and any sequence of distinct eigenvalues $(\lambda_n)$ in the spectrum of $T$ has a convergent subsequence.

Proof. Let $\lambda$ be a nonzero element of the spectrum of $T$. Then the operator $T - \lambda I$ is not bounded below. By Proposition 3.6, the range of $T - \lambda I$ is closed. Since $T$ is compact, its range is closed. Therefore, the range of $T - \lambda I$ is closed. This is the range of a compact operator. Hence, the range of $T - \lambda I$ is closed. Therefore, $\lambda$ is an eigenvalue of $T$.

Corollary 9.4.B. Let $X$ be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator.

Corollary 9.15. Let $X$ be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator. Then the range of $T - \lambda I$ is closed.

Theorem 9.15. Let $X$ be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator. Then the range of $T - \lambda I$ is closed.
Theorem 9.16 (continued)

Proof (continued).

Notice that for $a \leq k \leq n - 1$ we have

$$x^a - x^k = x^a(x^{k-a} - 1).$$