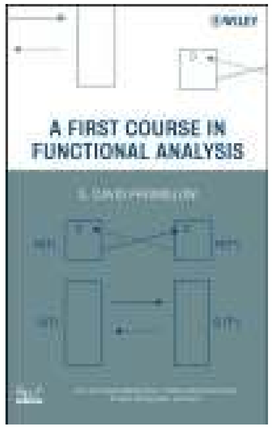
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.4. Spectrum of a Compact Operator—Proofs of Theorems



Lemma 9.13

Lemma 9.13. Let S be a linear operator from any linear space to itself. Consider the nested subsequences of subspace:
$R(S) \supset R(S^2) \supset R(S^3) \supset \cdots$ and $N(S) \subset N(S^2) \subset N(S^3) \subset \cdots$.

(a) Suppose that S is one to one (injective) but not onto (surjective). Then all inclusions in the range sequence are strict.

(b) Suppose that S is onto (surjective) but not one to one (injective). Then all inclusions in the null space sequence are strict.

Proof. (a) Suppose that S is one to one. If S is not onto, we can choose $x_1 \notin R(S)$. For this x_1, consider Sx_1. Certainly $Sx_1 \in R(S)$. ASSUME $Sx_1 \in R(S^2)$. Then $Sx_1 = S^2y$ for some y. But since S is one to one then this implies $x_1 = Sy$ and so $x_1 \in R(S)$, a CONTRADICTION. So $Sx \notin R(S^2)$ and $R(S^2)$ is a proper subset of $R(S)$. Similarly we can show there is some x_k with $S^kx_k \in R(S^k)$ but $S^kx_k \notin R(S^{k+1})$.

Lemma 9.13 (continued)

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$R(S) \supset R(S^2) \supset R(S^3) \supset \cdots$ and $N(S) \subset N(S^2) \subset N(S^3) \subset \cdots$.

(a) Suppose that S is one to one (injective) but not onto (surjective). Then all inclusions in the range sequence are strict.

(b) Suppose that S is onto (surjective) but not one to one (injective). Then all inclusions in the null space sequence are strict.

Proof (continued). (b) Suppose that S is onto. If S is not one to one, we can choose $z_1 \neq 0$ in $N(S)$ (recall that S is one to one if and only if $N(S) = \{0\}$; see page 5). Since S is onto, $x = Sx_1$ for some x_1. So $S^2x_1 = Sz = 0$ and $x_1 \in N(S^2)$ but $x_1 \notin N(S)$ since $Sx_1 = z \neq 0$. Similarly, we can show there is $x_k \in N(S^{k+1})$ but $x_k \notin N(S^k)$. □

Lemma 9.14

Lemma 9.14. Let X be a Banach space and let $Y \in \mathcal{B}(X)$ be compact. Let (λ_n) be a sequence of real or complex scalars. Suppose we have a strictly increasing sequence of (topologically) closed subspaces of X: $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$ such that $(T - \lambda_n I)Y_n \subset Y_{n-1}$ for all $n > 1$. Then $(|\lambda_n|)$ is not bounded below (in the sense given in Section 3.4). The same conclusion holds if we have a strictly decreasing sequence of closed subspaces of X: $Z_1 \supset Z_2 \supset Z_3 \supset \cdots$ such that $(T - \lambda_n I)Z_n \supset Z_{n+1}$ for all $n \in \mathbb{N}$.

Proof. We give a proof for the increasing case, with the decreasing case being similar. ASSUME (λ_n) is a sequence of scalars satisfying the stated conditions, where $(|\lambda_n|)$ is bounded below. Then there is $r > 0$ such that $|\lambda_n| \geq r$ for all $n \in \mathbb{N}$. By Theorem 2.33 (Riesz's Lemma) with $\varepsilon = 1/2$, for each $n \in \mathbb{N}$ there is a unit vector $y_n \in Y_n$ with $d(y_n, Y_{n-1}) \geq 1/2$. Consider the resulting sequence (of "almost perpendiculars") (y_n). For any $k < n$ we have

$$\|Ty_k - Ty_n\| = \| -\lambda_n y_n + \{(T - \lambda_k I)y_k - (T - \lambda_n I)y_n + \lambda_k y_k\}\|.$$

Lemma 9.14 (continued)

Proof (continued). We hypothesized that $(T-\lambda_n I)Y_n \subset Y_{n-1}$, so $(T-\lambda_n I)y_n \in Y_{n-1}$, since $k<n$ then $\lambda_k y_k \in Y_k \subset Y_{n-1}$, and $y_k \in Y_k \subset Y_n$ implies (by hypothesis) $(T-\lambda_k I)Y_k \in Y_{n-1}$. So $\{(T-\lambda_k I)y_k - (T-\lambda_n I)y_n + \lambda_k y_k\} \in T_{n-1}$ and $\{(T-\lambda_k I)y_k - (T-\lambda_n I)y_n + \lambda_k y_k\}/\lambda_n \in Y_{n-1}$. Hence by the choice of y_n we have

$$\begin{aligned}\|Ty_k - Ty_n\| &= |\lambda_n|\|y_n - \{(T-\lambda_k I)y_k - (T-\lambda_n I)y_n + \lambda_k y_k\}/\lambda_n\| \\ &\geq |\lambda_n| d(y_n, Y_{n-1}) \geq r/2.\end{aligned}$$

But the sequence $(Ty_n) \subset X$ cannot have a Cauchy subsequence and hence cannot have a convergent subsequence. But this contradicts the definition of compact operator (see page 187), CONTRADICTING the fact that T is compact. So the assumption that $(|\lambda_n|)$ is bounded below is false, and sequence (λ_n) satisfying the stated conditions must satisfy the fact that $(|\lambda_n|)$ is not bounded below. □

Corollary 9.4.A

Corollary 9.4.A. Let λ be a nonzero scalar and let $T \in \mathcal{B}(X)$ be compact where X is a Banach space. If $S = T - \lambda I$ is onto then it is one to one.

Proof. Consider the sequence of scalars (λ_n) where $\lambda_n = \lambda$ for all $n \in \mathbb{N}$. Then $(|\lambda_n|)$ is bounded below and with $Y_n = N(S^n)$ for $n \in \mathbb{N}$, we have $N(S) \subset N(S^2) \subset N(S^3) \subset \cdots$ (notice that these nullspaces are all topologically closed) and

$$(T-\lambda I)Y_n = SY_n = S(N(S^n)) \subset N(S^{n-1}) = Y_{n-1}$$

for all $n>1$ (since $x \in N(S^n)$ implies $S^n x = 0$, so $Sx \in N(S^{n-1})$ because $S^{n-1}(Sx) = S^n x = 0$). Therefore, Lemma 9.14 implies that the sequence $N(S) \subset N(S^2) \subset N(S^3) \subset \cdots$ cannot be strictly increasing. With S onto, by Lemma 9.13(b) we have that S must be one to one. □

Proposition 9.15

Proposition 9.15. Let X be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator. Then the range $R(T-I)$ is (topologically) closed. (This is the range of $R - \lambda I$ where $\lambda = 1$.)

Proof. Let $S = T - I$ and consider the quotient space $X/N(S)$. Then we have (see page 32) $S = S_1\pi$ where $\pi : X \to X/N(S)$ is the onto canonical projection $\pi(x) = x + N(S)$ and $S_1 : X/N(S) \to X$ is a onto to one mapping defined as $S_1(x+N(S)) = Sx$ ($S-1$ is denoted "$\tilde{T}$" on page 32).

First, we show that S_1 is bounded below (see Section 3.4). ASSUME S_1 is not bounded below. Then there is a sequence of unit vectors $(z_n) \subset X/N(S)$ such that $S_1 z_n \to 0$. Let $z_n = x_n + N(S)$. Then by the definition of the norm on $X/N(S)$, $\|z_n\| = d(z_n, N(S)) = \inf\{\|x_n - z\| \mid z \in N(S)\} = 1$. So fall all $\varepsilon > 0$ there is $z' \in N(S)$ such that $d(x_n, a) < 1 + \varepsilon$. In particular, with $\varepsilon = 1$, there is $z'_n \in N(S)$ such that $d(x_n, z'_n) = \|x_n - z'_n\| \leq 2$.

Proposition 9.15 (continued 1)

Proof (continued). Now $x_n - z'_n \in x + N(S)$ and $\pi(x_n - z'_n) = \pi(x_n) = z_n$. For for each $n \in \mathbb{R}$, there is $x'_n \in X$ (in our notation, $x'_n = x_n - z'_n$). Then by Lemma 9.1.B, the bounded sequence $\{x'_n\}$, the sequence (Tx'_n) has a convergent subsequence, say (Tx'_{n_k}). But also

$$\begin{aligned} Sx_n &= (S_1\pi)x_n \text{ since } S = S_1\pi \\ &= S_1 z_n \text{ since } \pi(x_n) = z_n, \end{aligned}$$

so that $Sx_{n_k} = S_1 z_{n_k} \to 0$. This means, since $S = T - I$,

$$x_{n_k} = Ix_{n_k} = (T-S)x_{n_k} = Tx_{n_k} - Sx_{n_k} \to z - 0 = x.$$

Since $S = T - I$ is bounded then it is continuous (by Theorem 2.6) and so $Sx_{n_k} \to Sx$, and so $Sx = 0$ and $x \in N(S)$. Hence $\pi(x) = 0$. By Theorem 2.27(c), $\|\pi\| = 1$ and so it is a bounded linear operator and hence is continuous (Theorem 2.6), so $z_{n_k} = \pi(x_{n_k}) \to \pi(x) = 0$. But each x_{n_k} is a unit vector (by choice) and a limit of unit vectors cannot equal the 0 vector (because the norm is continuous), a CONTRADICTION.

Proposition 9.15 (continued 2)

Proposition 9.15. Let X be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator. Then the range $R(T - I)$ is closed. (This is the range of $R - \lambda I$ where $\lambda = 1$.)

Proof (continued). So the assumption that S_1 is not bounded below. By Theorem 3.6, $R(S_1)$ is closed . Since $\pi : X \to X/N(S)$ is onto and $S = S_1\pi$, then $R(S) = R(S_1)$ and so $R(S) = R(T - I)$ is closed. □

Corollary 9.4.B

Corollary 9.4.B. Let λ be a nonzero scalar and let $T \in \mathcal{B}(X)$ be compact where X is a Banach space. If $S = T - \lambda I$ is one to one then it is onto.

Proof. Consider the sequence of scalars (λ_n) where $\lambda_n = \lambda$ for all $n \in \mathbb{N}$. Then $(|\lambda_n|)$ is bounded below and with $Y_n = R(S^n)$ for $n \in \mathbb{N}$ we have $R(S) \supset R(S^2) \supset R(S^3) \supset \cdots$ and each $R(S)$ is a (topologically) closed subspace of X by Proposition 9.15. So by Lemma 9.14 the sequence $R(S) \supset R(S^2) \supset R(S^3) \supset \cdots$ (with $Y_n = R(S^n)$) cannot be strictly increasing. With S one to one, by Lemma 9.13(a) we have that S must be onto. □

Theorem 9.16

Theorem 9.16. Let T be a compact operator in $\mathcal{B}(X)$, in which X is a Banach space. Then, the nonzero elements of the spectrum of T are eigenvalues. There are only countably many eigenvalues, and, in the case of infinitely many, they form a sequence tending to 0. The eigenspaces are all finite-dimensional.

Proof. Let $\lambda \neq 0$. If $T - \lambda I$ is one to one then it is onto by Corollary 9.4.B. So $T - \lambda I$ is then a bijection and hence is invertible. So λ cannot be in the spectrum then $T - \lambda I$ must not be invertible and hence (by definition) λ is in the point spectrum of T; that is, λ is an eigenvalue of T and the first claim holds.

Now we show the eigenspaces are finite dimensional. Suppose we have a sequence of vectors (x_n) of norm less than 1 in the eigenspace for eigenvalue λ. Since (x_n) is bounded and T is, by hypothesis, compact then by Corollary 9.1.B the sequence (Tx_n) has a convergent subsequence, say (Tx_{n_k}).

Theorem 9.16 (continued 1)

Proof (continued). Since the x_{n_k}'s are in the eigenspace, the convergent subsequence satisfies $(Tx_{n_k}) = \lambda x_{n_k})$. Since $(\lambda x_{n_k}) = \lambda(x_{n_k})$ converges then (x_{n_k}) converges. So arbitrary sequence (x_n) in the open unit ball of the eigenspace has a convergent subsequence, then by Corollary 9.1.A the open unit ball of the eigenspace is relatively compact. That is, $B(1)$ is a compact subset of the eigenspace. So by Reisz'z Theorem (Theorem 2.34) the eigenspace is finite dimensional.

We now consider the number and distribution of the eigenvalues. For any $r > 0$. Let $\mathcal{E}_r$ be the set of all eigenvalues of T with absolute value/modulus greater than r. ASSUME $\mathcal{E}_r$ is infinite. Then we can choose a sequence of distinct eigenvalues (λ_k) in $\mathcal{E}_r$. Let x_k be an eigenvector for λ_k and let $Y_n = \text{span}\{x_1, x_2, \ldots, x_n\}$. Since x_n is an eigenvector for λ_n then $(T - \lambda_n I)x_n = 0$.

Theorem 9.16 (continued 1)

Proof (continued).
Notice that for $a \leq k \leq n-1$ we have

$$(T - \lambda_n I)x_k = tx_k - \lambda_n x_k = \lambda_k x_k - \lambda_n x_k = (\lambda_k - \lambda_n)x_k \in Y_{n-1},$$

so $(T - \lambda_n I)Y_n \subset Y_{n-1}$. Promislow borrows a "basic fact" from linear algebra that a set of vectors for distinct eigenvalues is linearly independent. So $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$ is a strictly increasing sequence of closed subspaces (by Theorem 2.31(c), since Y_n is finite dimensional). Now $(\lambda_n) \subset \mathcal{E}_r$ so $|\lambda_n| \geq r$ for all $n \in \mathbb{N}$. But this implies that the sequence $|\lambda_n|$ is bounded below (see Section 3.4), CONTRADICTING Lemma 9.14. So the assumption that $\mathcal{E}_r$ is infinite is false and hence each $\mathcal{E}_r$ is finite. So the set of eigenvalues for T is $\cup_{n=1}^{\infty}\mathcal{E}_{1/m}$ (with 0 possibly included) and so is countable. If there are an infinite number of eigenvalues then for each $\varepsilon > 0$, the open ball centered at 0 with radius ε contains infinitely many eigenvalues (all but the finitely many in $\mathcal{E}_\varepsilon$) and hence the eigenvalues of T (no matter how enumerated) from a sequence with limit 0. □