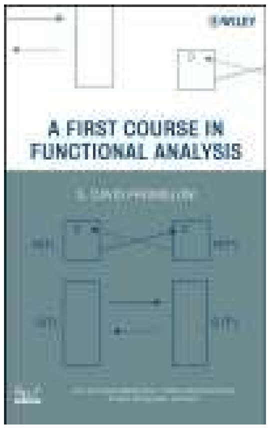
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.5. Compact Self Adjoint Operators on Hilbert Spaces—Proofs of Theorems



Proposition 9.17

Proposition 9.17. If M is invariant for compact, self adjoint operator T on a Hilbert space then $M^\perp$ is invariant for T. Moreover, the restrictions of T to both M and $M^\perp$ are also self adjoint.

Proof. For all $x \in M$ and $y \in M^\perp$ we have $\langle Ty, x\rangle = \langle y, T^*x\rangle = \langle y, Tx\rangle = 0$ since $Tx \in M$ because M is invariant under T. Therefore $Ty \in M^\perp$. Since y is an arbitrary element of $M^\perp$ then $M^\perp$ is invariant under T.

Since T is self adjoint on H and M and $M^\perp$ are invariant under T, the T restricted to M and $M^\perp$ is self adjoint (that is, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in M$ and for all $x, y \in M^\perp$). □

Theorem 9.18

Theorem 9.18. Spectral Theorem for Compact, Self Adjoint Operators.
Let T be a compact, self adjoint operator on a Hilbert space H. There is a sequence (either finite or countably infinite) of mutually orthogonal closed subspaces (M_n) whose closed linear span is all of H. There is a corresponding sequence (λ_n) of real numbers which if countably infinite converges to 0. For all n an d$x \in M_n$, we have $Tx = \lambda_n x$. Moreover, if $\lambda_n \neq 0$ then M_n is finite dimensional.

Proof. Let $\{\lambda_n\}$ be the set of distinct eigenvalues of T. Notice that each λ_n is real by Proposition 8.18(a). Let M_n be the eigenspace for λ_n (so $Tx = \lambda_n x$ for all $x \in M_n$). Let K be the closed span of all these eigenspaces: $K = \overline{\text{span}}\{M_n \mid n \text{ is an index for } \{\lambda_n\}\}$. Since the eigenvalues of $\{\lambda_n\}$ are distinct then the M_n are mutually orthogonal by Proposition 8.24. Since Hilbert space H is also a Banach space then by Theorem 9.16 each M_n is finite dimensional when $\lambda - n \neq 0$ (and so closed by Theorem 2.31(c)).

Theorem 9.18 (continued 1)

Proof (continued). If 0 is an eigenvalue then the corresponding eigenspace is the nullspace $N(T)$ which is closed since T is continuous. Also by Theorem 9.16, if there are a countably infinite number of eigenvalues then they converge to 0.

Now we show the final claim that $K = H$. Since each M_n is an eigenspace for λ_n, then M_n is invariant under T. So K is invariant under T (since each M_n is invariant and T is continuous on H by Theorem 2.6). Then by Proposition 9.17, $K^\perp$ is invariant under T. ASSUME $K^\perp \neq 0$. Let T_1 denote the restriction of T to $K^\perp$. Since a subset of any relatively compact set is relatively compact (the closure of the subset is a closed subset of the [compact] closure of the superset and so is compact; see page 18), from the definition of "compact operator" we have that the restriction of a compact operator must be compact. By Proposition 9.17, T_1 is self adjoint (on $K^\perp$). If T_1 is the zero operator on $K^\perp$, then there is some nonzero element x of $K^\perp$ mapped to 0 by T_1 and T.

Theorem 9.18 (continued 2)

Proof (continued). But then 0 is an eigenvalue for T and so x is in the eigenspace associated with eigenvalue 0 (it's one of the M_n's) and so $x \in K$, a contradiction since $K \cap K^\perp = \{0\}$ by the Projection Theorem [Theorem 4.14]); so T_1 is not the zero operator on $K^\perp$. By Proposition 8.21 either $\|T_1\|$ or $-\|T_1\|$ is in $\sigma(T_1)$. Since the value is nonzero, by Theorem 9.16 it is an eigenvalue of T_1, and so also is an eigenvalue of T. But then the corresponding (nonzero) eigenvectors is in both K an d$K^\perp$, a CONTRADICTION (again, by the Projection Theorem). So the assumption that $K^\perp \neq \{0\}$ is false, and $K^\perp = \{0\}$. That is, $H = K = \overline{\text{span}}\{M_n \mid n \text{ is an index for } \{\lambda_n\}\}$. □

Theorem 9.19

Theorem 9.19. For T a compact, self adjoint operator on Hilbert space H, $T = \sum_n \lambda_n E_{\lambda_n}$ in which E_{λ_n} is the projection onto M_n where M_n is the eigenspace associated with λ_n.

Proof. If T only has a finite number of eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, then H is the closed linear space of $M_1, M_2, \ldots, M_n$; that is, $H = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ (since there are only finitely many M_k's). But then for any $x \in H$, say $x = x_1 + x_2 + \cdots + x_n$ where $x_k \in M_k$, we have

$$\begin{aligned} T(x) &= T(x_1 + x_2 + \cdots + x_n) = T(x_1) + T(x_2) + \cdots + T(x_n) \\ &= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \\ &= \lambda_1 E_{\lambda_1}(x) + \lambda_2 E_{\lambda_2}(x) + \cdots + \lambda_n E_{\lambda_n}(x) \\ &= \sum_k \lambda_k E_{\lambda_k}, \end{aligned}$$

as claimed.

Theorem 9.19 (continued 1)

Proof (continued). If T has an infinite number of eigenvalues then, by the Spectral Theorem for Compact, Self Adjoint Operators (Theorem 9.18), the eigenvalues form a (countable) sequence (λ_n) with $(\lambda_n) \to 0$. Let $\varepsilon > 0$. Let $S_n = \sum_{k=1}^n \lambda_k E_{\lambda_k}$ (the nth partial sum) and let $T_n = T - S_n$ (the "tail"). Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Recall that a projection P satisfies (by definition) $P = P^*$ and $P^2 = P$, so the projection E_{λ_k} is self adjoint. By Proposition 9.10(a,b), T_n is self adjoint for all $n \in \mathbb{N}$. For $x \in M_k$ where $1 \leq k \leq n$ we have

$$\begin{aligned} T_n x &= (T - S_n)x = Tx - \sum_{k=1}^n \lambda_k E_{\lambda_k})x \\ &= Tx - \lambda_k E_{\lambda_k} x \text{ since } E_{\lambda_i} x = 0 \text{ for } i \neq k \\ &= \lambda_k x - \lambda_k x \text{ since } x \text{ is in eigenspace } M_k \text{ of } \lambda_k \\ &= 0. \end{aligned}$$

So T_n is 0 on $K = \overline{\text{span}}\{M_1, M_2, \ldots, M_n\}$ because T_n is continuous (since it is bounded; see Theorem 2.6).

Theorem 9.19 (continued 2)

Proof (continued). For $x \in K^\perp = \overline{\text{span}}\{M_1, M_2, \ldots, M_n\}^\perp$ we have

$$T_n x = (T - S_n)x = Tx - \sum_{k=1}^n \lambda_k E_{\lambda_k} x = Tx.$$

Next, if x is an eigenvector of T_n where $n \geq N$ with corresponding eigenvalue λ then

$$\lambda x = T_n x = T_x(x_K + x_{K^\perp}) = T_n x_{K^\perp} = T x_{K^\perp}$$

where $x_K \in K$ and $x_{K^\perp} \in K^\perp$. If $x \in K$ then $x_{K^\perp} = 0$. But then $\lambda x = T x_{K^\perp} = T0 = 0$ and so $\lambda = 0$. If $x \notin K$ then $x_{K^\perp} \neq 0$. Since $x \in H$ and H is the closed linear span of the M_n's, then $x = \sum_{k=1}^\infty a_k x_k$ where $x_k \in M_k$.

Theorem 9.19 (continued 3)

Proof (continued). Since T_n is continuous then

$$\begin{aligned} T_n x &= T_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{\infty} a_k T_n x_k \\ &= \sum_{k=n+1}^{\infty} a_k T_n x_k \text{ since } T_n \text{ is 0 on } M_1, M_2, \ldots, M_n \\ &= \sum_{k=n+1}^{\infty} a_k T x_k \text{ since all such } x_k \in K^{\perp} \\ &= \sum_{k=n+1}^{\infty} a_k \lambda_k x_k \text{ since } x_k \in M_k \\ &= \lambda x = \lambda \sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} a_k \lambda x_k. \end{aligned}$$

So $a_1 = a_2 = \cdots = a_n = 0$ and $a_k \lambda = a_k \lambda_k$ for $k \geq n+1$.

Theorem 9.19 (continued 4)

Proof (continued). Since $x_{K^\perp} \neq 0$ then some $a_k \neq 0$ for $k \geq n+1$ and then $\lambda = \lambda_k$ for some $k \geq n+1$. Since such λ_k satisfies $|\lambda_k| < \varepsilon$, then $|\lambda| < \varepsilon$. Therefore, any eigenvalue λ of T_n satisfies $|\lambda| < \varepsilon$ when $n > N$. Since T_n is compact, by Theorem 9.16 the nonzero elements of the spectrum are eigenvalues and so the spectral radius satisfies $r(T_n) < \varepsilon$. Since T_n is self adjoint, then $T_n = T_n^*$ and so $T_n T_N^* = T_n^* T_n$, so T_n is (by definition) normal. By Theorem 8.23, $\|T_n\| = r(T_n) < \varepsilon$. That is, for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ we have $\|T_n\| < \varepsilon$. So $(T_n) \to 0$ or $(T - S_n) \to 0$ or $S_n \to T$. That is,

$$T = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k E_{\lambda_k} = \sum_{n=1}^{\infty} \lambda_n E_{\lambda_n}.$$

□

Theorem 9.20

Theorem 9.20. A compact, self adjoint operator T on a separable Hilbert space is unitarily equivalent to a multiplication operator M_f on ℓ^2.

Proof. Choose an orthonormal basis of eigenvectors (e_n) and corresponding eigenvalues (μ_n) such that $T(x) = \sum_k \mu_k \langle x, e_k\rangle e_k$, as described in the note above. Let $U : \ell^2 \to H$ be defined as $U(\delta_n) = e_n$ where δ_n is the nth standard vector for ℓ^2. Then by Theorem 4.19 (see the proof of it) U is an isometric isomorphism (and so is bijective). Now

$$\begin{aligned} U^{-1}TU(\delta_n) &= U^{-1}T(e_n) = U^{-1}(\mu_n\langle e_n, e_n\rangle e_n) \\ &= U^{-1}(\mu_n e_n) = \mu_n U^{-1}(e_n) = \mu_n \delta_n. \end{aligned}$$

So with $f(x) = \mu_n$, then the multiplication operator M_f maps δ_n to $\mu_n\delta_n$. Since $U^{-1}TU$ and M_f agree on the basis $\{\delta_n\}_{n=1}^{\infty}$ of ℓ^2, then $U^{-1}TU$ and M_f are equal on ℓ^2. So M_f and T are (by definition) unitarily equivalent. □