

Introduction to Functional Analysis

Chapter 9. Compact Operators

9.6. Invariant Subspaces—Proofs of Theorems



Theorem 9.21

Theorem 9.21. Let X be a complex Banach space of dimension greater than 1. Any compact $T \in \mathcal{B}(X)$ has a closed proper invariant subspace.

Proof. If $T = 0$ then any closed proper subspace is invariant under T . So we assume $T \neq 0$. As commented in Note 2 above, if some \mathcal{P}_X is not dense for an $x \neq 0$ then the closure of \mathcal{P}_X is a closed proper invariant subspace of T . So we can also (without loss of generality) assume that \mathcal{P}_X is dense for all nonzero $x \in X$. Since $T \neq 0$, there is a unit vector $x_0 \in X$ such that $Tx_0 \neq 0$. Choose a neighborhood W of Tx_0 such that the closure \overline{W} does not contain 0. Choose a bounded neighborhood G of x_0 such that $T(G) \subset W$ (which can be done since T is continuous at x_0). Since x_0 is a unit vector, we can choose G such that all elements of G have norm greater than $1/2$ (or else take the given G and intersect it with the complement of the closed ball centered at 0 with radius 1.2).

Theorem 9.21 (continued 1)

Proof (continued). We make the following claim:

Claim. There is a positive constant k such that given any $n \in \mathbb{N}$ and any $x \in G$, we can find $p(T) \in \mathcal{P}$ with $\|p(T)\| \leq k^n$ such that $p(T)T^n \in G$.

We first give constant k and verify CLAIM for $n = 1$. Given any $t \in \overline{T(G)} \subset \overline{W}$, we know $y \neq 0$ (since $0 \notin \overline{W}$) and so we have assumed (without loss of generality) that \mathcal{P}_Y is dense in X and so \mathcal{P}_Y must intersect open set G . So there is $p_Y(T) \in \mathcal{P}$ such that $p_Y(T)y \in G$. Since $p_Y(T)$ is continuous at y , there is a neighborhood V_y of y such that $p_Y(T)(V_y) \subset G$. Since G is bounded (by choice) and T is compact then, by definition of “compact operator,” $\overline{T(G)}$ is relatively compact; that is, $\overline{T(G)}$ is compact. So for each $y \in \overline{T(G)}$ create a corresponding open set V_y containing y and a $p_Y(T) \in \mathcal{P}$ with these properties. Then $\{V_y \mid y \in \overline{T(G)}\}$ is an open cover of $\overline{T(G)}$ and since $\overline{T(G)}$ is compact, there are y_1, y_2, \dots, y_m such that $\overline{T(G)} \subset \bigcup_{i=1}^m V_{y_i}$.

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Theorem 9.21 (continued 2)

Proof (continued). Let $k = \max\{\|p_{y_i}(T)\| \mid i = 1, 2, \dots, m\}$. Then given $n = 1$ and any $x \in G$, we have $Tx \in T(G) \subset \overline{T(G)}$ and so $Tx \in V_{y_i}$ for some $i \in \{1, 2, \dots, m\}$ and $p_{y_i}(T)T^1x = p_{y_i}(T)Tx \in G$ (since $p_{y_i}(V_{y_i}) \subset G$) where $\|p_{y_i}(T)\| \leq k^1$. So CLAIM holds for $n = 1$.

We now prove CLAIM by induction. Notice that \mathcal{P} is closed under multiplication and all elements of \mathcal{P} commute with T . Suppose CLAIM holds for n . Then for given $x \in G$, we have $p_n(T) \in \mathcal{P}$ with $\|p_n(T)\| \leq k^n$ such that $p_n(T)T^n x \in G$. By considering $p_n(T)T^n x$ as a point in G (replacing x with $p_n(T)T^n x$ in CLAIM) then since CLAIM holds for $n = 1$, we have that there is $p_1(T) \in \mathcal{P}$ with $\|p_1(T)\| \leq k$ and $p_1(T)T^1(p_n(T)T^n x) = p_1(T)p_n(T)T^{n+1}x \in G$. So CLAIM holds with $p_{n+1}(T) = p_1(T)p_n(T)$. Notice that

$$\|p_1(T)p_n(T)\| \leq \|p_1(T)\| \|p_n(T)\| \leq k k^n = k^{n+1}$$

by Proposition 2.8. So CLAIM holds for $n + 1$.

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Theorem 9.21 (continued 3)

Proof (continued). Now for any $n \in \mathbb{N}$, let $x = x_0$ where unit vector $x_0 \in X$ with $Tx_0 \neq 0$ is from the first part of the proof. From CLAIM there is positive constant k and $p(T) \in \mathcal{P}$ such that $\|p(T)\| \leq k^n$ and $p(T)T^n x_0 \in G$. We then have (since all elements of G have norm greater than $1/2$)

$$\begin{aligned} \frac{1}{2} &< \|p(T)T^n x_0\| \leq \|p(T)\| \|T^n\| \|x_0\| \text{ by proposition 2.8} \\ &\text{and the definition of operator norm} \\ &\leq k^n \|T\|^n \text{ since } \|x_0\| = 1. \end{aligned}$$

This implies that

$$\|T^n\|^{1/n} \geq \frac{(1/2)^{1/n}}{k} \geq \frac{1}{2k}.$$

Now $\mathcal{B}(X)$ is a Banach algebra (see page 170), so by Theorem 8.15 the spectral radius if $r(T) = \inf \|T^n\|^{1/n} \geq 1/(2k)$. So the spectrum of T must contain a nonzero value. \square

Theorem 9.22

Theorem 9.22. Let X be a complex Banach space of dimension greater than 1. If $A \in \mathcal{B}(X)$ commutes with a nonzero compact operator T , then A has an invariant proper subspace.

Proof. If X is finite dimensional then every linear operator has only eigenvalues in its spectrum and so each eigenspace is an invariant subspace. So we can assume X is infinite dimensional without loss of generality.

Replace \mathcal{P} in the proof of Theorem 9.21 by $\mathcal{A} = \{S \in \mathcal{B}(X) \mid ST = TS\}$ (the collection of bounded operators which commute with T). Since A commutes with T by hypothesis, then for all $A \in \mathcal{A}$ we have $(AS)T = A(ST) = A(TS) = (AT)S = (T(AS))$ and so $AS \in \mathcal{A}$. \square

Theorem 9.21 (continued 4)

Theorem 9.21. Let X be a complex Banach space of dimension greater than 1. Any compact $T \in \mathcal{B}(X)$ has a closed proper invariant subspace.

Proof (continued). Since X is a Banach space and T is compact, by

Theorem 9.16 the nonzero element of the spectrum is an eigenvalue. Then (by Note 1) the eigenspace associated with this eigenvalue is a nontrivial invariant subspace and by Theorem 9.16 the eigenspace is finite dimensional and so (by Theorem 2.31(c)) the eigenspace is closed. If the eigenspace is all of X , then any proper subspace of X which exists since the dimension of X is greater than 1) is also invariant and the result follows. \square

Theorem 9.22 (continued)

Theorem 9.22. Let X be a complex Banach space of dimension greater than 1. If $A \in \mathcal{B}(X)$ commutes with a nonzero compact operator T , then A has an invariant proper subspace.

Proof (continued). So the argument in Note 2 above holds and we can assume that Ax is dense for all $x \in X$ as in the proof of Theorem 9.21. The same proof as given for Theorem 9.21 also carries through to show that T has a nonzero eigenvalue λ . Let M_λ be the associated eigenspace. Then $T(Ax) = ATx = A(\lambda x) = \lambda(Ax)$ for all $x \in M_\lambda$, that is $Ax \in M_\lambda$ and so M_λ is invariant under A . By Theorem 9.16, M_λ is finite dimensional and so is a proper subspace of X . \square