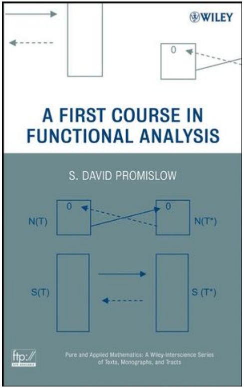
Introduction to Functional Analysis

Chapter 9. Compact Operators

9.6. Invariant Subspaces—Proofs of Theorems



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Theorem 9.21. Let X be a complex Banach space of dimension greater than 1. Any compact $T \in \mathcal{B}(X)$ has a closed proper invariant subspace.

Proof. If $T = 0$ then any closed proper subspace is invariant under T. So we assume $T \neq 0$. As commented in Note 2 above, if some $\mathcal{P}x$ is not dense for an $x \neq 0$ then the closure of $\mathcal{P}x$ is a closed proper invariant subspace of T. So we can also (without loss of generality) assume that $\mathcal{P}x$ is dense for all nonzero $x \in X$.

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Theorem 9.21 (continued 1)

Proof (continued). We make the following claim:

> **Claim.** There is a positive constant k such that given any $n \in \mathbb{N}$ and any $x \in G$, we can find $p(T) \in \mathcal{P}$ with $\|p(T)\| \leq k^n$ such that $p(T)T^n \in G$.

We first give constant k and verify CLAIM for $n = 1$. Given any $t \in \overline{T(G)} \subset \overline{W}$, we know $y \neq 0$ (since $0 \notin \overline{W}$) and so we have assumed (without loss of generality) that $\mathcal{P}y$ is dense in X and so $\mathcal{P}y$ must intersect open set G. So there is $p_y(T) \in \mathcal{P}$ such that $p_y(T)y \in G$.

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> **Claim.** There is a positive constant k such that given any $n \in \mathbb{N}$ and any $x \in G$, we can find $p(T) \in \mathcal{P}$ with $\|p(T)\| \leq k^n$ such that $p(T)T^n \in G$.

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Theorem 9.21 (continued 2)

Proof (continued). Let $k = \max\{\|p_{y_i}(T)\| \mid i = 1, 2, \ldots, m\}$. Then given $n = 1$ and any $x \in G$, we have $Tx \in T(G) \subset \overline{T(G)}$ and so $Tx \in V_{y_i}$ for some $i \in \{1, 2, \ldots, m\}$ and $p_{y_i}(T)T^1x = p_{y_i}(T)Tx \in G$ (since $p_{y_i}(V_{y_i}) \subset G$) where $\|p_{y_i}(T)\| \le k^1$. So CLAIM holds for $n = 1$.

We now prove CLAIM by induction. Notice that $\mathcal{P}$ is closed under multiplication and all elements of $\mathcal{P}$ commute with T. Suppose CLAIM holds for n. Then for given $x \in G$, we have $p_n(T) \in \mathcal{P}$ with $\|p_n(T)\| \le k^n$ such that $p_n(T)T^nx \in G$.

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$$\|p_1(T)p_n(T)\| \leq \|p_1(T)\|\|p_2(T)\| \leq kk^n = k^{n+1}$$

by Proposition 2.8. So CLAIM holds for $n + 1$.

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Proof (continued). Now for any $n \in \mathbb{N}$, let $x = x_0$ where unit vector $x_0 \in X$ with $Tx_0 \neq 0$ is from the first part of the proof. from CLAIM there is positive constant k and $p(T) \in \mathcal{P}$ such that $\|p(T)\| \leq k^n$ and $p(T)T^n x_0 \in G$. We then have (since all elements of G have norm greater than $1/2$)

$$\begin{aligned} \frac{1}{2} &< \|p(T)T^n x_0\| \leq \|p(T)\|\|T^n\|\|x_0\| \text{ by proposition 2.8} \\ &\quad \text{and the definition of operator norm} \\ &\leq k^n \|T\|^n \text{ since } \|x_0\| = 1. \end{aligned}$$

This implies that

$$\|T^n\|^{1/n} \geq \frac{(1/2)^{1/n}}{k} \geq \frac{1}{2k}.$$

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Proof (continued). Since X is a Banach space and T is compact, by Theorem 9.16 the nonzero element of the spectrum is an eigenvalue. Then (by Note 1) the eigenspace associated with this eigenvalue is a nontrivial invariant subspace and by Theorem 9.16 the eigenspace is finite dimensional and so (by Theorem 2.31(c)) the eigenspace is closed.

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Proof. If X is finite dimensional then every linear operator has only eigenvalues in it spectrum and so each eigenspace is an invariant subspace. So we can assume X is infinite dimensional without loss of generality.

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Proof (continued). So the argument in Note 2 above holds and we can assume that $\mathcal{A}x$ is dense for all $x \in X$ as in the proof of Theorem 9.21. The same proof as given for Theorem 9.21 also carries through to show that T has a nonzero eigenvalue λ. Let M_λ be the associated eigenspace. Then $T(Ax) = ATx = A(\lambda x) = \lambda(Ax)$ for all $x \in M_\lambda$, that is $Ax \in M_\lambda$ and so M_λ is invariant under A. By Theorem 9.16, M_λ is finite dimensional and so is a proper subspace of X. □

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