Lemma 5.1.1 (m = 1, Part I)

Proof (continued). If some coefficient $e_i^f \neq 0$, then we have the nontrivial solution.

variables $x_i$ from all equations (this system is obtained from the original one by eliminating the $i$-th equation). From the system of equations of the resulting system we have $m = 1 \leq \nu$.

If $\nu \geq m$, then we have the nontrivial solution $x_i^f = \ldots = x_i^m = 0$.

First, suppose we have $m \leq \nu$. Let $\nu = 1$.

Next suppose the result holds for a system of $\nu - 1$ equations in $\nu - 1$ unknowns.

This proves the result for $m = 1$ and $\nu$. 

Lemma 5.1.1 (m = 1, Part II)

Proof. We prove the result by induction on the number of equations. 

For some $i_1, i_2, \ldots, i_{\nu - 1}$, where $i_1 \neq i_2 \neq \ldots \neq i_{\nu - 1}$, then the system has a nontrivial solution (that is, a solution $x_1 = x_2 = \ldots = x_{\nu - 1}$). From this field $\mathbb{F}$, if $m < \nu$, then a system of $m$ equations in $\nu$ unknowns $x_1, x_2, \ldots, x_{\nu - 1}$ with coefficients $e_i$ and unknowns $x_j$ ($i \geq 1$, $j \leq \nu$), then we have the nontrivial solution $x_1 = x_2 = \ldots = x_{\nu - 1} = 0$.

\[ 0 = \sum_{i=1}^{\nu} e_i x_i + \sum_{j=1}^{\nu} e_j x_j + \sum_{k=1}^{\nu} e_k x_k 
\]

Lemma 5.1.1. Consider the homogeneous system of equations
Theorem 5.1.1 (continued)

For any $\mathbf{f} \in \mathbb{F}^n$ and $\mathbf{v} \in \mathcal{V}$, we have:

$$\langle \mathbf{f}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle$$

Since $\mathcal{V} = \{ \mathbf{v} \}$ is a linearly independent set, there is one-to-one:

$$\langle \mathbf{f}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle$$

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of $\langle \mathcal{A}, \mathcal{V} \rangle$. Define $A : \mathbf{v} \rightarrow \mathbf{v} \leftarrow \mathcal{A}$ as follows:

- Multiplication and vector addition are defined component-wise.
- $\mathcal{V} = \{ \mathbf{v} \mid \langle \mathbf{v}, \mathbf{v} \rangle = w \}$ where $\langle \mathcal{A}, \mathcal{V} \rangle = \mathcal{V}$ is isomorphic to
- $\mathcal{V}$ is an $n$-dimensional vector space, then $\mathbf{v}$ is isomorphic to
- Vector spaces.

Theorem 5.2.2 (Fundamental Theorem of Finite Dimensional Vector Spaces)

Theorem 5.1.1 (continued)

Proof (continued). The system of equations:

$$\begin{align*}
0 &= u_1w_1v_1 + \cdots + u_nv_nv_n + i_1w_1e_1 + \cdots + i_kw_ke_k + t_1w_1e_1 \\
0 &= u_1w_1v_1 + \cdots + u_nv_nv_n + i_1w_1e_1 + \cdots + i_kw_ke_k + t_1w_1e_1 \\
0 &= u_1w_1v_1 + \cdots + u_nv_nv_n + i_1w_1e_1 + \cdots + i_kw_ke_k + t_1w_1e_1 \\
\end{align*}$$

Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a vector space with bases $\langle \mathcal{A}, \mathcal{V} \rangle$.
we see that vector \( v \) is mapped equivalently under \( \hat{T} \) and \( A^T \).

\[
\begin{pmatrix}
\vec{u} \\
\vec{v} \\
\vec{w} \\
\vdots \\
\vec{u} \\
\vec{v} \\
\vec{w}
\end{pmatrix}
= A^T
\]

**Proof (continued).** Then defining

**Theorem 5.1.3.** If \( T \) is a linear transformation from \( n \)-dimensional vector space (\( \mathbb{R}^n \)) to \( m \)-dimensional vector space (\( \mathbb{R}^m \)) then \( T \) is equivalent to the action of an \( m \times n \) matrix \( A \).

\[
\begin{align*}
\text{then } T & \equiv \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \\
& \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \\
& \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix}
\end{align*}
\]

Therefore is an isomorphism.

\[
\begin{align*}
\varphi & = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \\
& \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix}
\end{align*}
\]

**Proposition 5.1.3.** Let \( \varphi \in \mathbb{R}^m \) and consider the representation of \( \varphi \) with respect to the standard basis of \( \mathbb{R}^m \).

\[
\begin{align*}
\varphi & = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \\
& \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix}
\end{align*}
\]

Then applying \( T \) to \( \varphi \) yields

\[
\begin{align*}
\varphi & = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \\
& \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix}
\end{align*}
\]