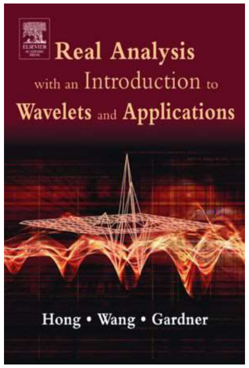
# Introduction to Functional Analysis

**Chapter 5. Vector Spaces, Hilbert Spaces, and the $L^2$ Space**

5.1. Groups, Fields, Vector Spaces—Proofs of Theorems








## Lemma 5.1.1

**Lemma 5.1.1.** Consider the homogeneous system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

with coefficients $a_{ij}$ $(1 \le i \le m,\ 1 \le j \le n)$ and unknowns $x_k$ $(1 \le k \le n)$ from field $\mathbb{F}$. If $n > m$ then the system has a nontrivial solution (that is, a solution $x_1, x_2, \ldots, x_n$ where $x_k \neq 0$ for some $1 \le k \le n$).

**Proof.** We prove the result by induction on the number of equations $m$. First, suppose we have $m = 1$ equation in $n > 1$ unknowns: $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$. If $a_{1j} = 0$ for $1 \le j \le n$, then we have the nontrivial solution $x_1 = x_2 = \cdots = x_n = 1$.





## Lemma 5.1.1 (continued 1)

**Proof (continued).** If some coefficient $a_{1j^*} \neq 0$, then we have the nontrivial solution

$$x_k = \begin{cases} 1 & \text{if} \quad k \neq j^* \\ -(a_{1j^*})^{-1}(a_{11} + a_{12} + \cdots + a_{1n} - a_{1j^*}) & \text{if} \quad k = j^* \end{cases}$$

This proves the result for $m = 1$ and $n > m$.

Next suppose the result holds for a system of $m - 1$ equations in $n - 1 > m - 1$ unknowns. If all coefficients $a_{ij} = 0$, then $x_1 = x_2 = \cdots = x_n = 1$ is a nontrivial solution.





## Lemma 5.1.1 (Continued 2)

**Proof (continued).** If some $a_{i^*j^*} \neq 0$, then consider the system of equations (this system is obtained from the original one by eliminating the variable $x_{j^*}$ from all equations):

$$\begin{array}{rcl} (a_{11} - (a_{i^*j^*})^{-1}a_{1j^*}a_{i^*1})x_1 & + & (a_{12} - (a_{i^*j^*})^{-1}a_{1j^*}a_{i^*2})x_2 + \cdots \\ & + & (a_{1j^*} - (a_{i^*j^*})^{-1}a_{1j^*}a_{i^*j^*})x_{j^*} + \cdots \\ & + & (a_{1n} - (a_{i^*j^*})^{-1}a_{1j^*}a_{i^*n})x_n = 0 \\ (a_{21} - (a_{i^*j^*})^{-1}a_{2j^*}a_{i^*1})x_1 & + & (a_{22} - (a_{i^*j^*})^{-1}a_{2j^*}a_{i^*2})x_2 + \cdots \\ & + & (a_{2j^*} - (a_{i^*j^*})^{-1}a_{2j^*}a_{i^*j^*})x_{j^*} + \cdots \\ & + & (a_{2n} - (a_{i^*j^*})^{-1}a_{2j^*}a_{i^*n})x_n = 0 \\ \vdots & \vdots & \vdots \\ (a_{m1} - (a_{i^*j^*})^{-1}a_{mj^*}a_{i^*1})x_1 & + & (a_{m2} - (a_{i^*j^*})^{-1}a_{mj^*}a_{i^*2})x_2 + \cdots \\ & + & (a_{mj^*} - (a_{i^*j^*})^{-1}a_{mj^*}a_{i^*j^*})x_{j^*} + \cdots \\ & + & (a_{mn} - (a_{i^*j^*})^{-1}a_{mj^*}a_{i^*n})x_n = 0 \end{array}$$

## Lemma 5.1.1 (continued 3)

**Proof (continued).** Notice that the coefficient of $x_{j^*}$ is 0 in each equation and that the $j^*$ equation is $0 = 0$. Therefore, this is a system of $m-1$ equations in the $n-1$ variables $x_1, x_2, \ldots, x_{j^*-1}, x_{j^*+1}, x_{j^*+2}, \ldots, x_n$. By the induction hypothesis, this system has a nontrivial solution, and this solution along with

$$x_{j^*} = -(a_{i^*j^*})^{-1}(a_{j^*1}x_1 + a_{j^*2}x_2 + \cdots + a_{j^*(j^*-1)}x_{j^*-1} + a_{j^*(j^*+1)}x_{j^*+1}$$
$$+ a_{j^*(j^*+2)}x_{j^*+2} + \cdots + a_{j^*n}x_n)$$

forms a nontrivial solution to the original system of equations. Hence, by induction, the result holds for all $m \geq 1$ and all $n > m$. □





## Theorem 5.1.1

**Theorem 5.1.1.** Let $\langle V, \mathbb{F}\rangle$ be a vector space with bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Then $n = m$.

**Proof.** Suppose $n > m$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a basis, then for some $a_{ij}$ where $1 \leq i \leq m$, $1 \leq j \leq n$ we have

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{m1}\mathbf{v}_m \\ \mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{m2}\mathbf{v}_m \\ \vdots &\ \ \vdots\ \ \vdots \\ \mathbf{w}_n &= a_{1n}\mathbf{v}_1 + a_{2n}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_m. \end{aligned}$$

Let $x_1, x_2, \ldots, x_n$ be ("unknown") elements of $\mathbb{F}$. Then

$$x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n = (x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n})\mathbf{v}_1 + (x_1a_{21} + x_2a_{22} +$$
$$\cdots + x_na_{2n})\mathbf{v}_2 + \cdots + (x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn})\mathbf{v}_m.$$





## Theorem 5.1.1 (continued)

**Theorem 5.1.1.** Let $\langle V, \mathbb{F}\rangle$ be a vector space with bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Then $n = m$.

**Proof (continued).** The system of equations

$$\begin{aligned} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} &= 0 \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} &= 0 \\ \vdots &\ \ \vdots\ \ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} &= 0 \end{aligned}$$

has a nontrivial solution $x_1, x_2, \ldots, x_n$ by Lemma 5.1.1, since $n > m$. Therefore $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n = \mathbf{0}$ for $x_1, x_2, \ldots x_n$ where $x_k \neq 0$ for some $1 \leq k \leq n$. That is, the set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is linearly dependent. But this is a contradiction since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis for $\langle V, \mathbb{F}\rangle$, and hence is a linearly independent set. Therefore $n \leq m$. Similarly, $m \leq n$ and we conclude that $n = m$. □





## Theorem 5.1.2

**Theorem 5.1.2 The Fundamental Theorem of Finite Dimensional Vector Spaces.**
If $\langle V, \mathbb{F}\rangle$ is an $n$-dimensional vector space, then $\langle V, \mathbb{F}\rangle$ is isomorphic to $\mathbb{F}^n = \langle V^*, \mathbb{F}\rangle$ where $V^* = \{(f_1, f_2, \ldots, f_n) \mid f_1, f_2, \ldots, f_n \in \mathbb{F}\}$, and scalar multiplication and vector addition are defined component wise.

**Proof.** Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of $\langle V, \mathbb{F}\rangle$. Define $\varphi : V \mapsto V^*$ as

$$\varphi((f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n)) = (f_1, f_2, \ldots, f_n).$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set, then $\varphi$ is one-to-one. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set of $\langle V, \mathbb{F}\rangle$ then $\varphi$ is onto. Finally, for any $f, f' \in \mathbb{F}$ and $\mathbf{v}, \mathbf{v}' \in V$ we have:

## Theorem 5.1.2 (continued)

**Proof (continued).**

$$\begin{aligned}
\varphi(f\mathbf{v} + f'\mathbf{v}') &= \varphi(f(f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n) + f'(f_1'\mathbf{v}_1 + f_2'\mathbf{v}_2 + \cdots \\
&\quad + f_n'\mathbf{v}_n)) \text{ where } \mathbf{v} = f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n \\
&\quad \text{ and } \mathbf{v}' = f_1'\mathbf{v}_1 + f_2'\mathbf{v}_2 + \cdots + f_n'\mathbf{v}_n \\
&= \varphi((ff_1 + f'f_1')\mathbf{v}_1 + (ff_2 + f'f_2')\mathbf{v}_2 + \cdots (ff_n + f'f_n')\mathbf{v}_n) \\
&= (ff_1 + f'f_1', ff_2 + f'f_2', \ldots, ff_n + f'f_n') \\
&= (ff_1, ff_2, \ldots, ff_n) + (f'f_1', f'f_2', \ldots, f'f_n') \\
&= f(f_1, f_2, \ldots, f_n) + f'(f_1', f_2', \ldots, f_n') \\
&= f\varphi(f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n) \\
&\quad + f'\varphi(f_1'\mathbf{v}_1 + f_2'\mathbf{v}_2 + \cdots + f_n'\mathbf{v}_n) \\
&= f\varphi(\mathbf{v}) + f'\varphi(\mathbf{v}').
\end{aligned}$$

Therefore $\varphi$ is an isomorphism. □





## Theorem 5.1.3

**Theorem 5.1.3.** If $T$ is a linear transformation from $n$-dimensional vector space $\langle V, \mathbb{F}\rangle$ to $m$-dimensional vector space $\langle W, \mathbb{F}\rangle$ then $T$ is equivalent to the action of an $m \times n$ matrix $A_T : \mathbb{F}^n \mapsto \mathbb{F}^m$.

**Proof.** Let $\mathbf{v} \in V$ and consider the representation of $\mathbf{v}$ with respect to the standard basis of $\langle V, \mathbb{F}\rangle$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n := (v_1, v_2, \ldots, v_n)$. Then applying $T$ to $\mathbf{v}$ yields

$$\begin{aligned}
T(\mathbf{v}) &= T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) \\
&= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n).
\end{aligned}$$

The vectors $T(\mathbf{e}_i)$, $1 \le i \le n$ are elements of $W$. Suppose that, with respect to the standard basis for $\langle W, \mathbb{F}\rangle$, we have the representation $T(\mathbf{e}_i) := (a_{1i}, a_{2i}, \ldots, a_{mi})$ for $1 \le i \le n$.





## Theorem 5.1.3 (continued)

**Theorem 5.1.3.** If $T$ is a linear transformation from $n$-dimensional vector space $\langle V, \mathbb{F}\rangle$ to $m$-dimensional vector space $\langle W, \mathbb{F}\rangle$ then $T$ is equivalent to the action of an $m \times n$ matrix $A_T : \mathbb{F}^n \mapsto \mathbb{F}^m$.

**Proof (continued.)** Then defining

$$A_T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we see that vector $\mathbf{v}$ is mapped equivalently under $T$ and $A_T$. □





## Theorem 5.1.4

**Theorem 5.1.4.** Let $\langle V, \mathbb{F}\rangle$ be a vector space. Then there exists a set of vectors $B \subset V$ such that (1) $B$ is linearly independent and (2) for any $\mathbf{v} \in V$ there exists finite sets $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subset B$ and $\{f_1, f_2, \ldots, f_n\}$ such that $\mathbf{v} = f_1\mathbf{b}_1 + f_2\mathbf{b}_2 + \cdots + f_n\mathbf{b}_n$. That is, $B$ is a Hamel basis for $\langle V, \mathbb{F}\rangle$.

**Proof.** Let $P$ be the class whose members are the linearly independent subsets of $V$. Then define the partial order $\prec$ on $P$ as $A \prec B$ for $A, B \in P$ if $A \subset B$. Now for $\mathbf{v} \neq \mathbf{0}$, $\{\mathbf{v}\} \in P$ and so $P$ is nonempty. Next, suppose $Q$ is a totally ordered subset of $P$. Define $M$ to be the union of all the sets in $Q$. Then $M \in P$ is an upper bound of $Q$. Hence by Zorn's Lemma, $P$ has a maximal element, call it $B$. Since $B$ is in $P$, $B$ is linearly independent. Also, any vector $\mathbf{v}$ must be a linear combination of elements of $B$, for if not, then the set $B \bigcup \{\mathbf{v}\}$ would be in $P$ and $B$ would not be maximal. □

## Exercise 5.1.3

**Exercise 5.1.3.** If $B_1$ and $B_2$ are Hamel bases for a given infinite dimensional vector space, then $B_1$ and $B_2$ are of the same cardinality.

**Proof.** Let $B_1 = \{\mathbf{b}_i \mid i \in I\}$. That is, let $B_1$ have indexing set $I$ so that $|B_1| = |I|$. For any $\mathbf{u} \in B_2$ we have that $\mathbf{u}$ is a finite linear combination of some finite subset of $B_1$, say

$$\mathbf{u} = \sum_{i \in J_{\mathbf{u}}} f_i \mathbf{b}_i$$

where $J_{\mathbf{u}} \subset I$ is a finite subset of set $I$. Now consider $J = \cup_{\mathbf{u} \in B_2} J_{\mathbf{u}}$. We have $J \subseteq I$ by construction.

ASSUME $J \neq I$. Then there is some $i' \in I$ where $i' \notin J$. Now $\mathbf{b}_{i'}$ is in the vector space and, since $B_2$ is a basis, then

$$\mathbf{b}_{i'} = \sum_{k \in K} f_k \mathbf{c}_k \text{ for some finite set } K \text{ and } \mathbf{c}_k \in B_2$$

where not all $f_k$ are 0.





## Exercise 5.1.3 (continued 1)

**Exercise 5.1.3.** If $B_1$ and $B_2$ are Hamel bases for a given infinite dimensional vector space, then $B_1$ and $B_2$ are of the same cardinality.

**Proof (continued).** We then have

$$\mathbf{b}_{i'} = \sum_{k \in K} f_k \left( \sum_{i \in J_{\mathbf{c}_k}} f_i \mathbf{b}_i \right) \text{ for some nonzero } f_i \in \mathbb{F}$$

(notice that, since $i' \notin J$, then $\mathbf{b}_{i'}$ is not in the sum on the right). But this implies that the finite set of vectors

$$\{\mathbf{b}_{i'}\} \cup \left\{ \mathbf{b}_i \,\middle|\, i \in J_{\mathbf{c}_k}, \mathbf{b}_{i'} = \sum_{k \in K} f_k \mathbf{c}_k \right\} \subset B_1$$

is not linearly independent, CONTRADICTING the fact that $B_1$ is a basis. So $J = I$.





## Exercise 5.1.3 (continued 2)

**Exercise 5.1.3.** If $B_1$ and $B_2$ are Hamel bases for a given infinite dimensional vector space, then $B_1$ and $B_2$ are of the same cardinality.

**Proof (continued).** Now each $J_{\mathbf{u}}$ is finite, and so by Exercise 0.8.11

$$|B_1| = |I| = |J| = |\cup_{\mathbf{u} \in B_2} J_{\mathbf{u}}| \le |B_2|.$$

Interchanging the roles of $B_1$ and $B_2$, we conclude that $|B_2| \le |B_1|$. Therefore, by the Schroeder-Bernstein Theorem (Theorem 0.8.6), $|B_1| = |B_2|$. □