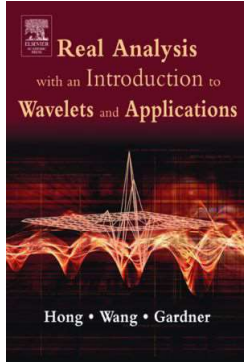


Introduction to Functional Analysis

Chapter 5. Vector Spaces, Hilbert Spaces, and the L^2 Space 5.2. Inner Product Spaces—Proofs of Theorems



Theorem 5.2.1

Theorem 5.2.1. The Schwarz Inequality.

For all \mathbf{u}, \mathbf{v} in inner product space $\langle V, \mathbb{C} \rangle$, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Proof. We know that for all $a \in \mathbb{C}$

$$\|\mathbf{u} + a\mathbf{v}\|^2 = \langle \mathbf{u} + a\mathbf{v}, \mathbf{u} + a\mathbf{v} \rangle \geq 0.$$

In particular, this inequality holds for $a = b \frac{\overline{\langle \mathbf{u}, \mathbf{v} \rangle}}{|\langle \mathbf{u}, \mathbf{v} \rangle|}$ where b is real. Therefore

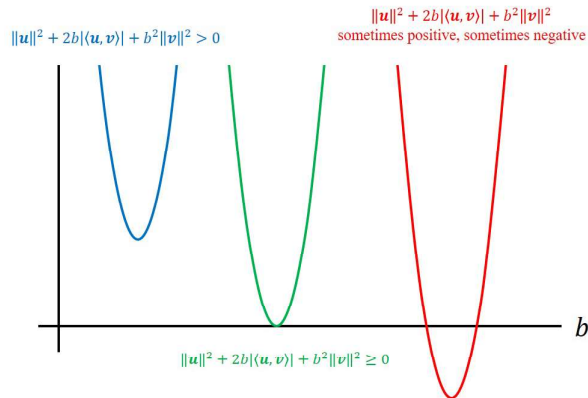
$$\begin{aligned} \langle \mathbf{u} + a\mathbf{v}, \mathbf{u} + a\mathbf{v} \rangle &= \|\mathbf{u}\|^2 + a\langle \mathbf{u}, \mathbf{v} \rangle + \overline{a\langle \mathbf{u}, \mathbf{v} \rangle} + |a|^2\|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + b \frac{\overline{\langle \mathbf{u}, \mathbf{v} \rangle}}{|\langle \mathbf{u}, \mathbf{v} \rangle|} \langle \mathbf{u}, \mathbf{v} \rangle + b \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\langle \mathbf{u}, \mathbf{v} \rangle|} \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + |b|^2\|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v} \rangle| + b^2\|\mathbf{v}\|^2 \geq 0. \end{aligned} \quad (1)$$

Theorem 5.2.1 (continued 1)

Proof (continued). Consider the function of b ,

$$f(b) = \|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v} \rangle| + b^2\|\mathbf{v}\|^2.$$

The graph of f is a concave-up parabola:



Theorem 5.2.1 (continued 2)

Theorem 5.2.1. The Schwarz Inequality.

For all \mathbf{u}, \mathbf{v} in inner product space $\langle V, \mathbb{C} \rangle$, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Proof (continued). To insure that (1) holds and that $\|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v} \rangle| + b^2\|\mathbf{v}\|^2 \geq 0$, we solve the equality

$$\|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v} \rangle| + b^2\|\mathbf{v}\|^2 = 0$$

for b , and get

$$b = \frac{-2|\langle \mathbf{u}, \mathbf{v} \rangle| \pm \sqrt{(2|\langle \mathbf{u}, \mathbf{v} \rangle|)^2 - 4\|\mathbf{v}\|^2\|\mathbf{u}\|^2}}{2\|\mathbf{v}\|^2}. \quad (2)$$

Therefore, inequality (1) holds for all \mathbf{u} and \mathbf{v} if and only if the discriminant of (2) is nonpositive:

$$(2|\langle \mathbf{u}, \mathbf{v} \rangle|)^2 - 4\|\mathbf{v}\|^2\|\mathbf{u}\|^2 \leq 0.$$

That is, $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$ □

Theorem 5.2.2. The Triangle Inequality

Theorem 5.2.2. The Triangle Inequality.

For all \mathbf{u}, \mathbf{v} in an inner product space $\langle V, \mathbb{C} \rangle$ we have $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Proof. We have

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + 2\operatorname{Re}\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ by the Schwarz Inequality} \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

Taking square roots yields the result. \square

Theorem 5.2.3. The Pythagorean Theorem

Theorem 5.2.3. The Pythagorean Theorem

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an orthonormal set of vectors in an inner product space $\langle V, \mathbb{C} \rangle$. Then for all $\mathbf{u} \in V$

$$\|\mathbf{u}\|^2 = \sum_{j=1}^n |\langle \mathbf{u}, \mathbf{v}_j \rangle|^2 + \left\| \mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\|^2.$$

Proof. Trivially

$$\mathbf{u} = \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right).$$

(We will see later that this is a rather fundamental decomposition of \mathbf{u} .)

Theorem 5.2.3 (Continued 1)

Proof (continued). Since $\left\langle \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle$

$$\begin{aligned} &= \left\langle \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \mathbf{u} \right\rangle - \left\langle \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle \\ &= \sum_{j=1}^n \overline{\langle \mathbf{v}_j, \mathbf{u} \rangle} \langle \mathbf{v}_j, \mathbf{u} \rangle - \sum_{j=1}^n \left\langle \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \sum_{k=1}^n \langle \mathbf{v}_k, \mathbf{u} \rangle \mathbf{v}_k \right\rangle \\ &= \sum_{j=1}^n \overline{\langle \mathbf{v}_j, \mathbf{u} \rangle} \langle \mathbf{v}_j, \mathbf{u} \rangle - \sum_{j=1}^n \sum_{k=1}^n \langle \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \langle \mathbf{v}_k, \mathbf{u} \rangle \mathbf{v}_k \rangle \\ &= \sum_{j=1}^n \overline{\langle \mathbf{v}_j, \mathbf{u} \rangle} \langle \mathbf{v}_j, \mathbf{u} \rangle - \sum_{j=1}^n \sum_{k=1}^n \overline{\langle \mathbf{v}_j, \mathbf{u} \rangle} \langle \mathbf{v}_k, \mathbf{u} \rangle \langle \mathbf{v}_j, \mathbf{v}_k \rangle \\ &= \sum_{j=1}^n |\langle \mathbf{v}_j, \mathbf{u} \rangle|^2 - \sum_{j=1}^n |\langle \mathbf{v}_j, \mathbf{u} \rangle|^2 = 0, \dots \end{aligned}$$

Theorem 5.2.3 (Continued 2)

Proof (continued). ... then these two vectors are orthogonal. Therefore

$$\begin{aligned} \|\mathbf{u}\|^2 &= \langle \mathbf{u}, \mathbf{u} \rangle = \left\langle \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right), \right. \\ &\quad \left. \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right) \right\rangle \\ &= \left\langle \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle \\ &\quad + \left\langle \mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle \\ &= \sum_{j=1}^n |\langle \mathbf{v}_j, \mathbf{u} \rangle|^2 + \left\| \mathbf{u} - \sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\|^2. \quad \square \end{aligned}$$