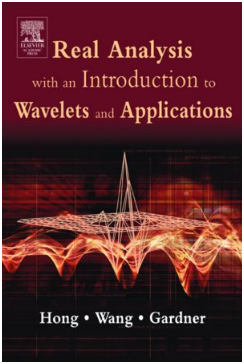
Introduction to Functional Analysis

Chapter 5. Vector Spaces, Hilbert Spaces, and the L^2 Space

5.2. Inner Product Spaces—Proofs of Theorems



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Theorem 5.2.1. The Schwarz Inequality.
For all $\mathbf{u}, \mathbf{v}$ in inner product space $\langle V, \mathbb{C} \rangle$, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \|\mathbf{v}\|.$$

Proof. We know that for all $a \in \mathbb{C}$

$$\|\mathbf{u} + a\mathbf{v}\|^2 = \langle \mathbf{u} + a\mathbf{v}, \mathbf{u} + a\mathbf{v} \rangle \ge 0.$$

In particular, this inequality holds for $a = b \dfrac{\overline{\langle \mathbf{u}, \mathbf{v} \rangle}}{|\langle \mathbf{u}, \mathbf{v} \rangle|}$ where b is real.

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Therefore

$$\begin{aligned}\langle \mathbf{u} + a\mathbf{v}, \mathbf{u} + a\mathbf{v}\rangle &= \|\mathbf{u}\|^2 + a\langle \mathbf{u}, \mathbf{v}\rangle + \overline{a\langle \mathbf{u}, \mathbf{v}\rangle} + |a|^2\|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + b\frac{\overline{\langle \mathbf{u}, \mathbf{v}\rangle}}{|\langle \mathbf{u}, \mathbf{v}\rangle|}\langle \mathbf{u}, \mathbf{v}\rangle + b\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{|\langle \mathbf{u}, \mathbf{v}\rangle|}\overline{\langle \mathbf{u}, \mathbf{v}\rangle} + |b|^2\|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v}\rangle| + b^2\|\mathbf{v}\|^2 \ge 0. \qquad (1)\end{aligned}$$

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Therefore

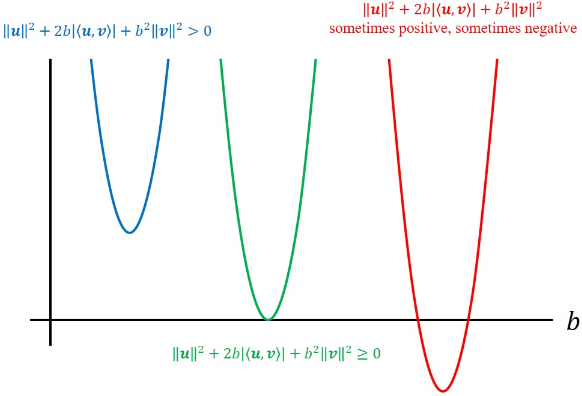
$$\begin{aligned}
\langle \mathbf{u} + a\mathbf{v}, \mathbf{u} + a\mathbf{v}\rangle &= \|\mathbf{u}\|^2 + a\langle \mathbf{u}, \mathbf{v}\rangle + \overline{a\langle \mathbf{u}, \mathbf{v}\rangle} + |a|^2\|\mathbf{v}\|^2 \\
&= \|\mathbf{u}\|^2 + b\frac{\overline{\langle \mathbf{u}, \mathbf{v}\rangle}}{|\langle \mathbf{u}, \mathbf{v}\rangle|}\langle \mathbf{u}, \mathbf{v}\rangle + b\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{|\langle \mathbf{u}, \mathbf{v}\rangle|}\overline{\langle \mathbf{u}, \mathbf{v}\rangle} + |b|^2\|\mathbf{v}\|^2 \\
&= \|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v}\rangle| + b^2\|\mathbf{v}\|^2 \geq 0. \qquad (1)
\end{aligned}$$

Theorem 5.2.1 (continued 1)

Proof (continued). Consider the function of b,

$$f(b) = \|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v}\rangle| + b^2\|\mathbf{v}\|^2.$$

The graph of f is a concave-up parabola:



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Proof (continued). To insure that (1) holds and that $\|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v}\rangle| + b^2\|\mathbf{v}\|^2 \geq 0$, we solve the equality

$$\|\mathbf{u}\|^2 + 2b|\langle \mathbf{u}, \mathbf{v}\rangle| + b^2\|\mathbf{v}\|^2 = 0$$

for b, and get

$$b = \frac{-2|\langle \mathbf{u}, \mathbf{v}\rangle| \pm \sqrt{(2|\langle \mathbf{u}, \mathbf{v}\rangle|)^2 - 4\|\mathbf{v}\|^2\|\mathbf{u}\|^2}}{2\|\mathbf{v}\|^2}. \qquad (2)$$

Therefore, inequality (1) holds for all $\mathbf{u}$ and $\mathbf{v}$ if and only if the discriminant of (2) is nonpositive:

$$(2|\langle \mathbf{u}, \mathbf{v}\rangle|)^2 - 4\|\mathbf{v}\|^2\|\mathbf{u}\|^2 \leq 0.$$

That is, $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$. □

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Theorem 5.2.2. The Triangle Inequality

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For all $\mathbf{u}, \mathbf{v}$ in an inner product space $\langle V, \mathbb{C} \rangle$ we have $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Proof. We have

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + \|\mathbf{v}\|^2 \\
&= \|\mathbf{u}\|^2 + 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ by the Schwarz Inequality} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}$$

Taking square roots yields the result. □

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Theorem 5.2.3. The Pythagorean Theorem

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Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set of vectors in an inner product space $\langle V, \mathbb{C}\rangle$. Then for all $\mathbf{u} \in V$

$$\|\mathbf{u}\|^2 = \sum_{j=1}^{n} |\langle \mathbf{u}, \mathbf{v}_j\rangle|^2 + \left\|\mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j\right\|^2.$$

Proof. Trivially

$$\mathbf{u} = \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j\right).$$

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(We will see latter that this is a rather fundamental decomposition of $\mathbf{u}$.)

Theorem 5.2.3 (Continued 1)

Proof (continued). Since $\left\langle \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j, \mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j \right\rangle$

$$\begin{aligned}
&= \left\langle \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j, \mathbf{u} \right\rangle - \left\langle \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j, \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j \right\rangle \\
&= \sum_{j=1}^{n} \overline{\langle \mathbf{v}_j, \mathbf{u}\rangle} \langle \mathbf{v}_j, \mathbf{u}\rangle - \sum_{j=1}^{n} \left\langle \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j, \sum_{k=1}^{n} \langle \mathbf{v}_k, \mathbf{u}\rangle \mathbf{v}_k \right\rangle \\
&= \sum_{j=1}^{n} \overline{\langle \mathbf{v}_j, \mathbf{u}\rangle} \langle \mathbf{v}_j, \mathbf{u}\rangle - \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j, \langle \mathbf{v}_k, \mathbf{u}\rangle \mathbf{v}_k \rangle \\
&= \sum_{j=1}^{n} \overline{\langle \mathbf{v}_j, \mathbf{u}\rangle} \langle \mathbf{v}_j, \mathbf{u}\rangle - \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{\langle \mathbf{v}_j, \mathbf{u}\rangle} \langle \mathbf{v}_k, \mathbf{u}\rangle \langle \mathbf{v}_j, \mathbf{v}_k \rangle \\
&= \sum_{j=1}^{n} |\langle \mathbf{v}_j, \mathbf{u}\rangle|^2 - \sum_{j=1}^{n} |\langle \mathbf{v}_j, \mathbf{u}\rangle|^2 = 0, \ldots
\end{aligned}$$

Theorem 5.2.3 (Continued 2)

Proof (continued). ...then these two vectors are orthogonal. Therefore

$$\begin{aligned}
\|\mathbf{u}\|^2 &= \langle \mathbf{u}, \mathbf{u} \rangle = \left\langle \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right), \right. \\
&\qquad \left. \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j + \left(\mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right) \right\rangle \\
&= \left\langle \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle \\
&\qquad + \left\langle \mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j, \mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\rangle \\
&= \sum_{j=1}^{n} |\langle \mathbf{v}_j, \mathbf{u} \rangle|^2 + \left\| \mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u} \rangle \mathbf{v}_j \right\|^2. \qquad \square
\end{aligned}$$