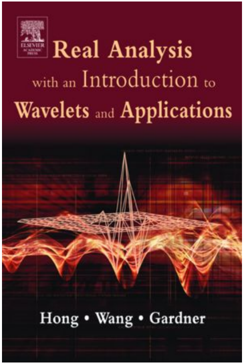
Introduction to Functional Analysis

Chapter 5. Vector Spaces, Hilbert Spaces, and the L^2 Space

5.4. Projections and Hilbert Space Isomorphisms—Proofs of Theorems



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Theorem 5.4.1

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$$\langle h, s\rangle = \left\langle \lim_{n\to\infty} s_n, s \right\rangle = \lim_{n\to\infty} \langle s_n, s\rangle = 0$$

since the inner product is continuous (Exercise 6 of Section 5.2). So $h \in S^\perp$ and (s_n) converges in $S^\perp$. Therefore $S^\perp$ is complete. □

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Theorem 5.4.2

Theorem 5.4.2. Let S be a subspace of a Hilbert space H (that is, the set of vectors in S is a subset of the set of vectors in H, and S itself is a Hilbert space). Then for any $h \in H$, there exists a unique $t \in S$ such that $\inf_{s \in S} \|h - s\| = \|h - t\|$.

Proof. Let $d = \inf_{s \in S} \|h - s\|$ and choose a sequence $(s_n) \subset S$ such that $\lim_{n \to \infty} \|h - s_n\| = d$. Then

$$\begin{aligned}
\|s_m - s_n\|^2 &= \|(s_m - h) - (s_n - h)\|^2 \\
&= 2\|s_m - h\|^2 + 2\|s_n - h\|^2 - \|s_m + s_n - 2h\|^2 \\
&\quad \text{by the Parallelogram Law (Exercise 3 of Section 5.2)} \\
&= 2\|s_m - h\|^2 + 2\|s_n - h\|^2 - 4\left\|h - \frac{1}{2}(s_m + s_n)\right\|^2 \\
&\le 2\|s_m - h\|^2 + 2\|s_n - h\|^2 - 4d^2 \text{ since } \frac{1}{2}(s_m + s_n) \in S.
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Theorem 5.4.2 (continued)

Proof (continued). Now the fact that $\lim_{n\to\infty} \|h - s_n\| = d$, implies that as $m, n \to \infty$, $\|s_m - s_n\| \to 0$ and so (s_n) is Cauchy and hence convergent to some $t_1 \in S$, where $\lim_{n\to\infty} \|h - s_n\| = \|h - t_1\|$. For uniqueness, suppose for some $t_2 \in S$ we also have $\lim_{n\to\infty} \|h - s_n\| = \|h - t_2\|$. Then

$$\begin{aligned} \|t_1 - t_2\|^2 &= 2\|h - t_1\|^2 + 2\|h - t_2\|^2 - 4\left\|h - \frac{1}{2}(t_1 + t_2)\right\|^2 \text{ (as above)} \\ &= 4d^2 - 4\left\|h - \frac{1}{2}(t_1 + t_2)\right\|^2. \end{aligned}$$

Next, $\frac{1}{2}(t_1 + t_2) \in S$ and so $\left\|h - \frac{1}{2}(t_1 + t_2)\right\| \geq \inf_{s\in S} \|h - s\| = d$.
Therefore $\|t_1 - t_2\|^2 = 0$ and $t_1 = t_2$. □

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Theorem 5.4.3

Theorem 5.4.3. Let S be a subspace of a Hilbert space H. Then for all $h \in H$, there exists a unique decomposition of the form $h = s + s'$ where $s \in S$ and $s' \in S^{\perp}$.

Proof. For $h \in H$, let $t \in S$ be as defined in Theorem 5.4.2. Let $r = h - t$. We will show $s' = r \in S^{\perp}$. Now for any $s_1 \in S$ and any scalar a,

$$\|r\|^2 = \|h - t\|^2 \leq \|h - (t + as_1)\|^2 = \|r - as_1\|^2$$

$$= \langle r - as_1, r - as_1 \rangle = \|r\|^2 - \langle as_1, r \rangle - \langle r, as_1 \rangle + |a|^2 \|s_1\|^2.$$

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Therefore $0 \leq |a|^2 \|s_1\|^2 - \langle as_1, r \rangle - \langle r, as_1 \rangle$. If the inner product is complex valued and a is real, then we have $0 \leq a^2 \|s_1\|^2 - 2a\mathrm{Re}\langle r, s_1 \rangle$. If $a \in \mathbb{R}$ with $a > 0$, then with $a \to 0^+$ we have that $\mathrm{Re}\langle r, s_1 \rangle \leq 0$. If $a \in \mathbb{R}$ with $a < 0$, then with $a \to 0^-$ we have that $\mathrm{Re}\langle r, s_1 \rangle \geq 0$. So $\mathrm{Re}\langle r, s_1 \rangle = 0$.

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Theorem 5.4.3 (continued)

Proof (continued). If a is purely imaginary, say $a = ib$ where $b \in \mathbb{R}$, then

$$\begin{aligned} 0 &\leq b^2\|s_1\|^2 - \langle ibs_1, r\rangle - \langle r, ibs_1\rangle \\ &= b^2\|s_1\|^2 + ib\langle s_1, r\rangle - ib\langle r, s_1\rangle \\ &= b^2\|s_1\|^2 + ib(2i\mathrm{Im}\langle s_1, r\rangle) \\ &= b^2\|s_1\|^2 - 2b\mathrm{Im}\langle s_1, r\rangle. \end{aligned}$$

Therefore, $b^2\|s_1\|^2 \geq 2b\mathrm{Im}\langle s_1, r\rangle$ and similar to above, by considering $b > 0$, $b \to 0^+$ and $b < 0$, $b \to 0^-$ we see that $\mathrm{Im}\langle s_1, r\rangle = 0 = \mathrm{Im}\langle r, s_1\rangle$. Hence, $\langle r, s_1\rangle = 0$ for all $s_1 \in S$, and therefore $r \in S^\perp$. So we have written h as $h = t + r$ where $t = s \in S$ and $r = s' \in S^\perp$.

Now suppose that $h = t_1 + r_1 = t_2 + r_2$ where $t_1, t_2 \in S$ and $r_1, r_2 \in S^\perp$. Then $t_1 - t_2 = r_2 - r_1$ where $t_1 - t_2 \in S$ and $r_2 - r_1 \in S^\perp$. Since the only element common to S and $S^\perp$ is 0, then $t_1 - t_2 = r_2 - r_1 = 0$ and $t_1 = t_2$ and $r_1 = r_2$. Therefore the decomposition of h is unique. □

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Theorem 5.4.4

Theorem 5.4.4. A Hilbert space with a Schauder basis has an orthonormal basis.

Proof. We start with a Schauder basis $S = \{s_1, s_2, \ldots\}$ and construct an orthonormal basis $R = \{r_1, r_2, \ldots\}$ using a method called the *Gram-Schmidt process*. First define $r_1 = s_1/\|s_1\|$.

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$$r_k = \frac{s_k - \text{proj}_{R_k}(s_k)}{\|s_k - \text{proj}_{R_k}(s_k)\|}.$$

Then for $i \neq j$,

$$\langle r_i, r_j \rangle = \frac{\left\langle s_i - \text{proj}_{R_i}(s_i), s_j - \text{proj}_{R_j}(s_j) \right\rangle}{\|s_i - \text{proj}_{R_i}(s_i)\| \|s_j - \text{proj}_{R_j}(s_j)\|}.$$

Clearly the r_i's are unit vectors. We leave as an exercise the proof that the r_i's are pairwise orthogonal. □

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$$h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k.$$

Proof. We know from Bessel's Inequality (Corollary 5.2.1) that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} |\langle h, r_k \rangle|^2 \leq \|h\|^2.$$

Therefore $s_n = \sum_{k=1}^{n} |\langle h, r_k \rangle|^2$ form a monotone bounded sequence of real numbers and hence converges and is Cauchy. Define $h_n = \sum_{k=1}^{n} \langle h, r_k \rangle r_k$.

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Theorem 5.4.5 (continued)

Proof (continued). Then for $n > m$, by the Pythagorean Theorem (Theorem 5.2.3),

$$\|h_n - h_m\|^2 = \left\| \sum_{k=m+1}^{n} \langle h, r_k \rangle r_k \right\|^2 = \sum_{k=m+1}^{n} |\langle h, r_k \rangle|^2,$$

and as a consequence, the sequence (h_n) is a Cauchy sequence in H and so converges to some $h' \in H$. So for each $i \in \mathbb{N}$,

$$\begin{aligned} \langle h - h', r_i \rangle &= \left\langle h - \lim_{n\to\infty} \sum_{k=1}^{n} \langle h, r_k \rangle r_k, r_i \right\rangle \\ &= \lim_{n\to\infty} \left\langle h - \sum_{k=1}^{n} \langle h, r_k \rangle r_k, r_i \right\rangle \\ &= \langle h, r_i \rangle - \langle h, r_i \rangle = 0 \end{aligned}$$

Therefore by Exercise 8 in the text, $h - h' = 0$ and $h = \sum^{\infty} \langle h, r_k \rangle r_k$. □

Theorem 5.4.6

Theorem 5.4.6. Let $\{r_1, r_2, \ldots\}$ be an orthonormal basis for Hilbert space H, let $R_k = \text{span}\{r_1, r_2, \ldots, r_{k-1}\}$, and let $h \in H$. Then $\inf_{s \in R_k} \|h - s\| = \|h - t\|$ where $t = \sum_{i=1}^{k-1} \langle h, r_i \rangle r_i$. That is, best approximations of h are given by partial sums of the power series of h.

Proof. ASSUME to the contrary that there exists $t' \in R_k$ where $\inf_{s \in R_k} \|h - s\| = \|h - t'\| < \|h - t\|$ (we know the infimum holds for a unique element of R_k by Theorem 5.4.3). Then $t' = \sum_{i=1}^{k-1} t_i' r_i$ for some $t_1', t_2', \ldots, t_{k-1}'$. As we will see in Theorem 5.4.7,

$$\|h - t\|^2 = \left\| \sum_{i=k}^{\infty} \langle h, r_i \rangle r_i \right\|^2 = \sum_{i=k}^{\infty} |\langle h, r_i \rangle|^2.$$

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$$\|h - t\|^2 = \left\| \sum_{i=k}^{\infty} \langle h, r_i \rangle r_i \right\|^2 = \sum_{i=k}^{\infty} |\langle h, r_i \rangle|^2.$$

Theorem 5.4.6 (continued)

Proof (continued). Also

$$\begin{aligned} \|h - t'\|^2 &= \left\| \sum_{i=1}^{k-1} (\langle h, r_i \rangle - t_i') r_i + \sum_{i=k}^{\infty} \langle h, r_i \rangle r_i \right\|^2 \\ &= \sum_{i=1}^{k-1} |\langle h, r_i \rangle - t_i'|^2 + \sum_{i=k}^{\infty} |\langle h, r_i \rangle|^2. \\ &= \sum_{i=1}^{k-1} |\langle h, r_i \rangle - t_i'|^2 + \|h - t\|^2 \geq \|h - t\|^2. \end{aligned}$$

Clearly this CONTRADICTS $\|h - t'\| < \|h - t\|$. So the assumption that such a t' exists is false, and the claim follows. □

Theorem 5.4.8

Theorem 5.4.8 A Hilbert space with scalar field $\mathbb{R}$ or $\mathbb{C}$ is separable if and only if it has a countable orthonormal basis.

Proof. Suppose H is separable and $D = \{d_1, d_2, \ldots\}$ is dense in H. For $k \geq 2$ define $D_k = \text{span}\{d_1, d_2, \ldots, d_{k-1}\}$ and $e_k = d_k - \text{proj}_{D_k}(d_k)$. Then the set $E = \{e_1, e_2, \ldots\} \setminus \{0\}$ is linearly independent (in the sense of Schauder). The set of all finite linear combinations of elements of E are dense in H (since this includes all elements of D; notice that the first $k-1$ elements of E span D_{k-1}). So E is a spanning set of H (in the sense of Schauder). Normalizing the elements of E gives an orthonormal basis of H, as claimed.

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Theorem 5.4.8 (continued 1)

Theorem 5.4.8 A Hilbert space with scalar field $\mathbb{R}$ or $\mathbb{C}$ is separable if and only if it has a countable orthonormal basis.

Proof (continued). Next, suppose $R = \{r_1, r_2 \ldots\}$ is an orthonormal basis for H. Then for each $h \in H$, $h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k$, by Theorem 5.4.5. Let $\varepsilon > 0$ be given. By Theorem 5.4.7, $\|h\|^2 = \sum_{k=1}^{\infty} |\langle h, r_k \rangle|^2$, and so there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\sum_{k=n}^{\infty} |\langle h, r_k \rangle|^2 < \varepsilon/2$. For each $k \in \{1, 2, \ldots, N\}$, there exists a rational number (or a rational complex number if $\langle h, r_k \rangle$ is complex) a_k such that $|\langle h, r_k \rangle - a_k|^2 < \frac{\varepsilon}{2^{N+1}}$. Then

$$\left\| h - \sum_{k=1}^{N} a_k r_k \right\|^2 = \left\| \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k - \sum_{k=1}^{N} a_k r_k \right\|^2$$

$$= \left\| \sum_{k=1}^{N} (\langle h, r_k \rangle r_k - a_k r_k) + \sum_{k=N+1}^{\infty} \langle h, r_k \rangle r_k \right\|^2$$

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Proof (continued).

$$= \left\| \sum_{k=1}^{N} (\langle h, r_k\rangle r_k - a_k r_k) \right\|^2 + \left\| \sum_{k=N+1}^{\infty} \langle h, r_k\rangle r_k \right\|^2 \text{ by the Pythagorean Theorem (Theorem 5.2.3)}$$

$$= \sum_{k=1}^{N} |\langle h, r_k\rangle - a_k|^2 + \sum_{k=N+1}^{\infty} |\langle h, r_k\rangle|^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and the set

$$D = \{\textstyle\sum_{k=1}^{n} a_k r_k \mid n \in \mathbb{N} \text{ and } a_k \text{ is rational (or complex rational)}\}$$

is a countable dense subset of H. □

Theorem 5.4.9. The Fundamental Theorem of Infinite Dimensional Vector Spaces

Theorem 5.4.9. The Fundamental Theorem of Infinite Dimensional Vector Spaces.
Let H be a Hilbert space with a countable infinite orthonormal basis. Then H is isomorphic to ℓ^2.

Proof. Let the orthonormal basis of H be $R = \{r_1, r_2, \ldots\}$. Then for $h \in H$, define $\pi(h)$ to be the sequence of inner products of h with the elements of R: $\pi(h) = (\langle r_1, h\rangle, \langle r_2, h\rangle, \ldots)$. Then π is linear:

$$\begin{aligned}
\pi(ah_1 + bh_2) &= (\langle r_1, ah_1 + bh_2\rangle, \langle r_2, ah_1 + bh_2\rangle, \ldots) \\
&= (a\langle r_1, h_1\rangle + b\langle r_1, h_2\rangle, a\langle r_2, h_1\rangle + b\langle r_2, h_2\rangle, \ldots) \\
&= a\left(\langle r_1, h_1\rangle, \langle r_2, h_1\rangle, \ldots\right) + b\left(\langle r_1, h_2\rangle, \langle r_2, h_2\rangle, \ldots\right) \\
&= a\pi(h_1) + b\pi(h_2).
\end{aligned}$$

By Theorem 5.4.5, $h = \sum_{k=1}^{\infty} \langle h, r_k\rangle r_k$ and by Theorem 5.4.7 $\pi(h) \in \ell^2$.

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$$\begin{aligned} \pi(ah_1 + bh_2) &= (\langle r_1, ah_1 + bh_2\rangle, \langle r_2, ah_1 + bh_2\rangle, \ldots) \\ &= (a\langle r_1, h_1\rangle + b\langle r_1, h_2\rangle, a\langle r_2, h_1\rangle + b\langle r_2, h_2\rangle, \ldots) \\ &= a\left(\langle r_1, h_1\rangle, \langle r_2, h_1\rangle, \ldots\right) + b\left(\langle r_1, h_2\rangle, \langle r_2, h_2\rangle, \ldots\right) \\ &= a\pi(h_1) + b\pi(h_2). \end{aligned}$$

By Theorem 5.4.5, $h = \sum_{k=1}^{\infty} \langle h, r_k\rangle r_k$ and by Theorem 5.4.7 $\pi(h) \in \ell^2$.

Theorem 5.4.9 (continued 1)

Theorem 5.4.9. The Fundamental Theorem of Infinite Dimensional Vector Spaces.
Let H be a Hilbert space with an infinite orthonormal basis. Then H is isomorphic to ℓ^2.

Proof (continued). Now the representation of h in terms of the basis elements is unique, so π is one-to-one. Next, let $(a_1, a_2, \ldots) \in \ell^2$, and consider the partial sums, s_n, of $\sum_{k=1}^{\infty} a_k r_k$. Then for $m < n$,
$\|s_n - s_m\|^2 = \left\|\sum_{k=m+1}^{n} a_k r_k\right\|^2 = \sum_{k=m+1}^{n} |a_k|^2$. Since $\displaystyle\sum_{k=1}^{\infty} |a_k|^2$ converges (its associated sequence of partial sums is a monotone, bounded sequence of real numbers), then the sequence of partial sums of this series is convergent and hence Cauchy. Therefore, (s_n) is a Cauchy sequence in H and hence is convergent. So π is onto.

Theorem 5.4.9 (continued 2)

Proof (continued). Now consider $h, h' \in H$ where $h = \sum_{k=1}^{\infty} a_k r_k$ and $h' = \sum_{k=1}^{\infty} a'_k r_k$. Then

$$\begin{aligned} \langle h, h' \rangle &= \left\langle \sum_{k=1}^{\infty} a_k r_k, \sum_{k=1}^{\infty} a'_k r_k \right\rangle = \sum_{k=1}^{\infty} a_k a'_k \\ &= \langle (a_1, a_2, \ldots), (a'_1, a'_2, \ldots) \rangle = \langle \pi(h), \pi(h') \rangle . \end{aligned}$$

Therefore π is a Hilbert space isomorphism and H is isomorphic to ℓ^2, as claimed. □