Introduction to Functional Analysis, MATH 5740

Homework 1, Chapter 1

Due Thursday, June 11 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

1.1(a). Let s denote the linear space consisting of all sequences over field $\mathbb{F}$. Let c_{00} denote the set of all sequences that have only finitely many nonzero entries. Prove that c_{00} is an infinite dimensional subspace of s. Find a basis of c_{00}. HINT: Suppose $\{b_1, b_2, \ldots, b_n\}$ is a basis and define m_k to be the position in sequence b_k corresponding to the last nonzero entry.

1.3(a). Given any $y \in s$, define a function $M_y : s \to s$ by $M_y(x)(n) = y(n)x(n)$. That is, M_y multiplies each entry of x by the corresponding entry of y (it is like an entry-wise product). Prove that M_y is a linear operator. HINT: Perform computations entry-wise (i.e., in terms of n) but draw general conclusions about elements of s (i.e., sequences).

1.3(b). Describe the spaces $N(M_y)$ (the nullspace of M_y) and $R(M_y)$ (the range of M_y).

1.4(a). Define functions S and T from s (the linear space consisting of all sequences from field $\mathbb{F}$) to itself by:

$$S(x(1), x(2), x(3), \ldots) = (0, x(1), x(2), x(3), \ldots),$$

$$T(x(1), x(2), x(3), \ldots) = (x(2), x(3), x(4), \ldots).$$

S is called a *right shift* and T is called a *left shift*. Prove that S and T are linear operators.

1.7. Suppose that for each λ in some index set Λ we are given a subspace Y_λ of linear space X such that for all $\lambda, \mu \in \Lambda$ we have that either $Y_\lambda \subseteq Y_\mu$ or $Y_\mu \subseteq Y_\lambda$. Prove that $\cup_{\lambda \in \Lambda} Y_\lambda$ is a subspace of X. HINT: From Linear Algebra, we know that we need only show that $\cup Y_\lambda$ is closed under vector addition and scalar multiplication. So show that for $y_1, y_2 \in \cup Y_\lambda$ and $\alpha \in \mathbb{F}$, we have $y_1 + y_2 \in \cup Y_\lambda$ and $\alpha y_1 \in \cup Y_\lambda$.

1.8. **(Bonus)** Let S be an infinite set, and let X be a linear space of functions that includes δ_s for all $s \in S$ where δ_s is the function that takes on the value 1 at s and is 0 elsewhere. Prove that X is infinite dimensional.