Introduction to Functional Analysis, MATH 5740

Homework 2, Chapter 2

Due Tuesday, June 16 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

2.2(a). Prove part (a) of Theorem 2.3: Suppose that $(x_n)$and (y_n) are sequences in a normed linear space, (α_n) is a sequence in $\mathbb{F}$, that $x = \lim(x_n)$, $y = \lim(y_n)$, and $\alpha = \lim(\alpha_n)$. Prove $\lim(x_n + y_n) = \lim(x_n) + \lim(y_n) = x + y$.

2.6. Suppose that $\|\cdot\|$ is a seminorm on X. A seminorm satisfies all the properties of a norm EXCEPT: $\|x\| = 0 \Rightarrow x = 0$.

- **(a)** Prove that $S = \{x \in X \mid \|x\| = 0\}$ is a subspace of X. HINT: From Linear Algebra, we know that we need only show that S is closed under vector addition and scalar multiplication.
- **(b)** Prove that $\|x - y\| = 0$ implies that $\|x\| = \|y\|$.

2.9. Prove that if Y is a subspace of a normed linear space X, then Y is closed if and only if the set $A = \{y \in Y \mid \|y\| \leq 1\}$ is closed. HINT: $A \subseteq Y$ is closed if and only if $A = \overline{A}$. Use part (iii) of the Theorem about $\overline{A}$ on page 6 of the class notes for Section 2.2.

2.10. Prove that a linear operator $T : X \to Z$, where X and Z are linear spaces, is completely determined by its values on $B(x_0, r)$ for any $x_0 \in X$ and for any $r > 0$. HINT: Stretch/shrink x and translate x until it is in $B(x_0, r)$. Apply T to the result and solve for $T(x)$.