Introduction to Functional Analysis, MATH 5740

Homework 4, Chapter 2

Due Wednesday, June 24 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

2.20. Consider the sup norm on c_0 and c_{00}.

- **(a)** Prove that in c_0 the series $\sum_{k=1}^{\infty} \delta_k/k$ is convergent but not absolutely convergent.
- **(b)** Give a series in c_{00} that is absolutely convergent but not convergent. HINT: Think p-series. NOTICE: By Theorem 2.12 that this implies that c_{00} is not complete.

2.22. Prove that for linear space X with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, if for all $r > 0$ there is $s > 0$ such that $B_2(0; s) \subseteq B_1(0; r)$ (where B_i denotes a ball with respect to $\|\cdot\|_i$), then $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. HINT: Use the definition of weaker norm.

2.27. Prove that if a normed linear space X has a complete subspace M such that X/M is complete, then X is complete. HINT: Use the completeness of M to show M is closed. Choose an absolutely summable series in X, $\sum_{k=1}^{\infty} \|x_k\|$. Show $\sum_{k=1}^{\infty} \|\overline{x}_k\|$ is absolutely summable. Use this WITH DETAILS to show that $\sum_{k=1}^{\infty} x_k$ is summable and use Theorem 2.12.

2.25. (Bonus) Let X be the space of all continuous functions f on $[0, 1]$ such that $f(0) = 0$, with the sup norm. Let $Y = \{f \in X \mid \int_0^1 f(t)\,dt = 0\} \subseteq X$. Prove that for any $g \in X$ and positive integer $n \in \mathbb{N}$, there is a constant $\alpha \in \mathbb{R}$ such that $g - \alpha t^{1/n} \in Y$. Conclude that perpendiculars to closed proper subspaces need not exist. HINT: Show that Y is closed (you will need uniform convergence under the sup norm for this). Assume g is a perpendicular to Y. Use properties of $\|g\|_\infty$ and $\int_0^1 g(t)\,dt$ to conclude g is the 0 function.