Introduction to Functional Analysis, MATH 5740

Homework 6, Chapter 3

Due Monday, July 6 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

3.2. In c_{00} find a nested sequence of closed sets with diameters approaching zero that has an empty intersection. HINT: Define $x_n = (1 + 1/n, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$ and $x_{-n} = (1 - 1/n, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$. Let $E_N = \{x_n, x_{-n} \mid n \in \mathbb{N}, n \geq N\}$. Show that the E_N are nested, closed, and $\operatorname{diam}(E_N) \to 0$, but $\cap E_N = \varnothing$.

3.3. Find a sequence $y \in \ell^\infty$ such that the multiplication operator M_y on ℓ^∞ is injective (one to one) but not bounded below. HINT: Let y be any element of c_0 which is not in c_{00}.

3.4. Let X be a linear space that has a countably infinite Hamel basis. Prove that there is no norm on X that will make it into a Banach space. Conclude that there is no norm on c_{00} that will make it a Banach space. HINT: Let $B = \{b_1, b_2, \ldots\}$ be a Hamel basis. Define $V_n = \operatorname{span}\{b_1, b_2, \ldots, b_n\}$ and show that $X = \cup_{n=1}^{\infty} V_n$. Show that each V_n has a nonempty interior in X (but every point of X is an interior point of X, trivially). Use Corollary 3.3.